SUMMARIES OF THE PUBLICATIONS

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Included in the documentation of the procedure for the academic position "professor"
in the field of higher education 4. Natural science, Mathematics and Informatics,
professional field 4.5. Mathematics (Mathematical Analysis)
(the publications hereby are not part of any other publications or research works required for the scientific PhD title, Dr. Sci., and the academic position of Associate Professor)

1 Books

1. O. Christov, S. Hakkaev, Lecture notes on ordinary differential equations, (2013) ISBN 978-954-577-956-5

These notes were written for students at Shumen University and cover the material for students in Mathematics and Informatics, Informatics and Computer Science.

The Authors have tried to make these notes self-contained as possible. All theoretical parts are supported by several examples. Besides solved examples, a number of exercise questions are given at the end of every section. The book covers the topics as: first order differential equations, second order linear differential equations, linear systems and qualitative methods. In addition, a brief discussion about basic topics from Mathematical Analysis and Linear Algebra are also given.

2 Publications

1. **J. Angulo, A. Corcho, S. Hakkaev**, Well-posedness and stability in the periodic case for the Benney system, **Advances in Differential Equations**, 16(2011), 523-550 (Q1)

In this paper we consider the system introduced by Benny which models the interaction between short and long waves

$$\begin{cases} iu_t + u_{xx} = uv + \beta|u|^2u, \\ v_t = (|u|^2)_x, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1}$$

where $u(x,t)$ is a complex-valued function representing the short wave and $v(x,t)$ is a real-valued function representing the long wave. Denote, by H^s_{per} standard Sobolev space of periodic functions and introduce modified Boourgain space

$$||f||_{X^{s,b}_{per}} = (\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} (1+|n|)^{2s}(1+|\tau+n^2|^{2b}|\widehat{f}(n,\tau)|^2)d\tau)^{\frac{1}{2}},$$

$$||f||_{X^r_{per}} = ||f||_{X^{r,\frac{1}{2}}_{per}} + ||\langle n\rangle^r \widehat{f}(n,\tau)||_{l^2_n L^1_\tau},$$

$$||f||_{Y^s_{per}} = ||f||_{H_t^{\frac{1}{2}} H^s_{per}} + ||\langle n\rangle^s \widehat{f}(n,\tau)||_{l^2_n L^1_\tau}.$$

Concerning local well-posedness we obtain the following result.

Theorem 2.1. *For any $(u_0,v_0) \in H^r_{per} \times H^s_{per}$ with r,s satisfying the condition*

$$max\{0, r-1\} \leq s \leq min\{r, 2r-1\}$$

there exist a positive time $T = T(||u_0||_r, ||v_0||_s)$ and a unique solution $(u(x,t), v(x,t))$ of the initial value problem (1), satisfying

$$(a)\ (\eta_T(t)u, \eta_T(v)) \in X^r_{per} \times Y^s_{per}$$
$$(b)\ (u,v) \in C(\triangle T; H^r_{per} \times H^s_{per}).$$

Moreover, the map $(u_0,v_0) \to (u(t),v(t))$ is locally uniformly continuous from $H^r_{per} \times H^s_{per}$ into $C(\triangle T; H^r_{per} \times H^s_{per})$.

Also we find a region for which the Cauchy problem is not locally well-posed.

Theorem 2.2. *Let $\beta \neq 0$. Then, for any $r < 0$ and $s \in \mathbb{R}$, the initial value problem (1)is locally ill posed in $H^r_{per} \times H^s_{per}$.*

Regarding the stability of periodic traveling waves, namely solutions for (1) of the form

$$\begin{cases} u(x,t) = e^{-iwt}e^{ic(x-ct)/2}\varphi_{w,c}(x-ct) \\ v(x,t) = n_{w,c}(x-ct), \end{cases}$$

where $\varphi_{w,c}, n_{w,c}$ are real, smooth, L-periodic functions, first we find explicit form for the functions in Jacobian elliptic form, namely

$$\begin{cases} \varphi_{w,c}(x-ct)\sqrt{\frac{c}{1-\beta c}}\eta_1 dn(\frac{\eta_1}{\sqrt{2}};\kappa) \\ n_{w,c}(x-ct) = \frac{\eta_1^2}{1-\beta c}dn^2(\frac{\eta_1}{\sqrt{2}};\kappa) \end{cases} \tag{2}$$

and then we obtain the following stability theorem.

Theorem 2.3. *Let* $(w,c) \in \mathcal{A}_\beta$*, where* $\mathcal{A}_\beta = \{(x,y) : y > 0, 1 > \beta y,\ x < -\frac{-2\pi^2}{L^2} - \frac{y^2}{4}\}$ *such that for* $c > 0$ *there is* $q \in \mathbb{N}$ *satisfying* $3\pi q/c = L$*. Define* $\sigma = -w - \frac{c^2}{4}$*. Then* $(\Phi(\xi) = e^{ic\xi/2}\varphi_{w,c}(\xi), \Psi(\xi) = n_{w,c}(\xi)$*, with* $\varphi_{w,c}, n_{w,c}$ *given by (2), is orbitally stable in* $H^1_{per}([0,L]) \times L^2_{per}([0,L])$*:*

(a) for $\beta \leq 0$

(b) for $\beta > 0$ *and* $8\beta\sigma - 3c(1-\beta c)^2 \leq 0$.

2. **O. Christov, S. Hakkaev, I. Iliev,** Non-uniform continuity of periodic Holm-Staley b-family of equations, **Nonlinear Analysis**, 75 (2012), 4821-4838 (Q1)

We consider a family of non-evolutionary partial differential equations known as Holm - Staley b - family which includes the integrable Camassa-Holm and Degasperis-Procesi equations.

$$m_t + um_x + bu_xm = 0, \tag{3}$$

with $m = u - u_{xx}$, $u(x,t)$ representing the fluid velocity, while the constant b is a balance or a bifurcation parameter for the solution behavior. We show that the solution map is not uniformly continuous. The proof relies on a construction of smooth periodic travelling waves with small amplitude.Our main result in this paper is the following.

Theorem 2.4. *For any* $s \geq 3$*, the solution map* $u_0 \to u$ *for the equation (3) with* $b \neq 0$*, is not uniformly continuous from any bounded set in* $\mathrm{H}^s(\mathbb{S})$ *into*

$C([0,t_0], H^s(\mathbb{S}))$. *More precisely, for each $s \geq 3$ there exist constants $c_{1,2} > 0$ and two sequences of smooth solutions u_n, v_n of the equation (3) such that for any $t \in [0,1]$*

$$\sup_n ||\, u_n(t)||_{H^s} + \sup_n ||v_n(t)||_{H^s} \leq c_1,$$

$$\lim_{n\to\infty} ||\, u_n(0) - v_n(0)||_{H^s} = 0,$$

$$\liminf_n ||\, u_n(t) - v_n(t)||_{H^s} \geq c_2 \sin\left(\frac{t}{2}\right).$$

3. **S. Hakkaev. M. Stanislavova, A. Stefanov**, Transverse instability for periodic traveling waves of KP-I and Schrodinger equations, **Indiana University Mathematics Journal**, 61(2)(2012), 461-492 (Q3)

We consider the quadratic and cubic KP - I

$$\begin{cases} (u_t + \partial_{xxx}u + \partial_x(f(u)))_x - \partial_{yy}u = 0, \quad (t,x,y) \in \mathbb{R}_+ \times [0,K_1] \times [0,K_2] \\ u(t,x+K_1,y) = u(t,x,y); u(t,x,y+K_2) = u(t,x,y) \end{cases} \tag{4}$$

and NLS models in $1+2$ dimensions with periodic boundary conditions

$$\begin{cases} iu_t - (u_{xx} + u_{yy}) - f(|u|^2)u = 0, (t,x,y) \in \mathbb{R}_+ \times [0,K_1] \times [0,K_2] \\ u(t,x+K_1,y) = u(t,x,y);\; u(t,x,y+K_2) = u(t,x,y). \end{cases} \tag{5}$$

In this paper we only deal with the stability information provided by the linearized equation. Suppose that the linearized equation is in the form of an evolution equation

$$v_t = \mathcal{A}v. \tag{6}$$

We use the following definition of spectral and linear stability

Definition 1. Assume that $\mathcal{A} = \mathcal{A}(\varphi)$ generates a C_0 semigroup on a Banach space X. We say that the solution φ with linearized problem (6) is spectrally stable, if $\sigma(\mathcal{A}) \subset \{\lambda : \Re\lambda \leq 0\}$.

We say that the the solution φ with linearized problem (6) is linearly stable, if the growth bound for the semigroup $e^{t\mathcal{A}}$ is non-positive. Equivalently, we require that every solution of (6) with $v(0) \in X$ has the property

$$\lim_{t\to\infty} e^{-\delta t}\|v(t,\cdot)\| = 0$$

for every $\delta > 0$.

We consider periodic traveling waves of the KdV and mKdV equation, which in turn also solve the KP-I equation. In order to explain the stability/instability results, we need to linearize the equation (4) about the periodic traveling wave solution. Namely, write an ansatz in the form $u(t,x,y) = \varphi(x-ct) + v(t,x-ct,y)$, which we plug in (4). After ignoring all nonlinear in v terms, we arrive at the following linear equation for v

$$(v_t + v_{xxx} - cv + (f'(\varphi)v)_x)_x - \partial_{yy}v = 0. \tag{7}$$

In order to establish instability, we seek solutions in the form

$$v(t,x,y) = e^{\sigma t}e^{iky}V(x),$$

where $\sigma \in \mathbb{C},\ k \in \mathbb{R}$ and $V(x)$ is periodic function with same period as the periodic traveling wave solution $\varphi(x)$.

We further specialize V in the form $V = \partial_x U$. Plugging yields the equation

$$-\sigma\partial_x U = (-\partial_x(-\partial_{xx} + c - f'(\varphi))\partial_x + k^2)U.$$

This eigenvalue problem is therefore in the form

$$\sigma A(k)U = L(k)U, \tag{8}$$

with

$$A(k) = -\partial_x,\ \ L(k) = -\partial_x(-\partial_{xx} + c - f'(\varphi))\partial_x + k^2.$$

where $L(k), A(k)$ are operators which depend on the real parameter k on some Hilbert space H. Our first result concerns the transverse instability of the cnoidal solutions of the KP - I equation.

Theorem 2.5. *Considering the KP - I equation (i.e. (4) with $f(u) = \frac{u^2}{2}$). Then, there exists a period K_2 depending on the particular cnoidal solution, so that the cnoidal waves are spectrally and linearly unstable for all values of the parameters $\kappa \in (0,1)$ and T.*

Next, we state our main result regarding transverse instability of the dnoidal solutions of the modified KP - I equation.

Theorem 2.6. *Consider the modified KP - I equation, that is (4) with $f(u) = u^3$. Then, there exists a period K_2 depending on the particular dnoidal solution, so that the dnoidal solutions are spectrally and linearly unstable for all values of the parameters $\kappa \in (0,1)$ and the corresponding T.*

Our second example concerns spatially periodic standing waves of the non-linear Schrödinger equation (NLS). This allows us to write the linearized problem as follows

$$\sigma \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 & \mathcal{L}_+ + k^2 \\ -(\mathcal{L}_- + k^2) & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{9}$$

Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Note that $J^* = J^{-1} = -J$. In terms of z_1, z_2, we have the equation

$$\sigma J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -J \begin{pmatrix} \mathcal{L}_- + k^2 & 0 \\ 0 & \mathcal{L}_+ + k^2 \end{pmatrix} J \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Thus, we have managed to recast the problem in the form (8), this time with

$$A(k) = \sigma J; \;\; L(k) = -J \begin{pmatrix} \mathcal{L}_- + k^2 & 0 \\ 0 & \mathcal{L}_+ + k^2 \end{pmatrix} J = -J \begin{pmatrix} \mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ \end{pmatrix} J + k^2 Id \tag{10}$$

we have the following result, which shows transverse instability for standing waves of the quadratic and cubic NLS. That is, we shall be considering (5) with $f(z) = \sqrt{z}$ and $f(z) = z$.

Theorem 2.7. *The quadratic (focussing) Schrödinger equation* (5) *admits cnoidal solutions. There exists K_2, depending on the specific solution, so that these solutions are spectrally and linearly unstable for all values of the parameter $\kappa \in (0, 1)$.*

The cubic (focussing) Schrödinger equation (5) *supports dnoidal solutions. There exists K_2, depending on the specific solution, so that these solutions are spectrally and linearly unstable for all values of the parameter $\kappa \in (0, 1)$.*

4. **S. Hakkaev, M. Stanislavova, A. Stefanov**, Orbital stability for periodic standing waves for the Klein-Gordon-Zakharov system and the beam equation, **Zeitschrift fuer Angewandte Mathematik und Physik**, 64(2) (2013), 265-282 (Q1)

The existence and stability of spatially periodic waves $(e^{i\omega t}\varphi_\omega, \psi_\omega)$ in the Klein-Gordon-Zakharov (KGZ) system

$$\begin{cases} u_{tt} - u_{xx} + u + uv = 0, \\ v_{tt} - c^2 v_{xx} = (|u|^2)_{xx} \end{cases} \tag{11}$$

are studied. First we show a local existence result for low regularity initial data.

Theorem 2.8. *Let $\alpha > 1/2$. Then, the Cauchy problem for* (11)*,* **considered both for** $x \in \mathbf{R}$ **or in the periodic context** $0 < x < 1$ *is locally well-posed in the space $H^\alpha \times H^{\alpha-1} \times H^{\alpha-1} \times H^{\alpha-2}$.*

Then, we construct a one-parameter family of periodic dnoidal waves for (KGZ) system when the period is biger than $\sqrt{2}\pi$. We show that these waves are stable whenever an appropriate function satisfies the standard Grillakis-Shatah-Strauss type condition.

Theorem 2.9. *Let $L > \sqrt{2}\pi$ be a given period. Then, there is orbital stability for all ω satisfying the inequality*

$$\sqrt{-\frac{G(\kappa_0(L))}{F(\kappa_0(L))}} \leq |\omega| \leq \sqrt{1 - \frac{2\pi^2}{L^2}} \tag{12}$$

where

$$F(\kappa) = \left[2(2-\kappa^2)E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)(1-\kappa^2)K^2(\kappa)\right]$$
$$G(\kappa) = 2(1-\kappa^2)E(\kappa)K(\kappa) - (2-\kappa^2)E^2(\kappa)$$

where $E(\kappa), K(\kappa)$ are the standard elliptic functions, $\kappa_0(L)$ is the inverse function to the increasing function

$$\kappa \to \frac{2\sqrt{2-\kappa^2}K(\kappa)}{\sqrt{1+\frac{G(\kappa)}{F(\kappa)}}}, \quad \kappa \in (0,1)$$

We compute the intervals for the parameter ω explicitly in terms of L and by taking the limit $L \to \infty$ we recover the previously known stability results for the solitary waves in the whole line case.

For the beam equation

$$u_{tt} + \Delta^2 u + u - |u|^{p-1}u = 0, \tag{13}$$

we show the existence of spatially periodic standing waves and show that orbital stability holds if an appropriate functional satisfies Grillakis-Shatah-Strauss type condition.

5. **S. Hakkaev, I. Iliev, K. Kirchev**, Stability of periodic traveling waves for the quadratic and cubic nonlinear Schrodinger equations, **International Journal of Bifurcation and Chaos**, 23(5)(2013), 1350090 (20pp.) (Q2)

In this work we consider the nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + |u|^p u = 0. \tag{14}$$

Our principal aim is to study the the orbital stability of the family of periodic traveling-wave solutions

$$u = \varphi(x,t) = e^{i(vx+(\omega-v^2)t)} r(x-2vt). \tag{15}$$

where $r(y)$ is a real-valued T-periodic function and $v, \omega \in \mathbf{R}$ are parameters, for quadratic (p=1) and cubic (p=2) nonlinear Schrödinger equation.

We base our analysis on some appropriate invariant laws. Our approach is to verify that φ is a minimizer of a properly chosen functional M which is conservative with respect to time over the solutions of (14). We consider the L^2-space of T-periodic functions in $x \in \mathbf{R}$, with a norm $||.||$ and a scalar product $\langle .,. \rangle$. To establish that the orbit

$$\mathcal{O} = \{e^{i\eta}\varphi(\cdot - \xi, t) : \ (\xi,\eta) \in [0,T] \times [0,2\pi]\}$$

is stable, we take

$$u(x,t) = e^{i\eta}\varphi(x-\xi,t) + h(x,t) = e^{i\zeta}[r(x-\xi-2vt) + h_1 + ih_2]$$

and express the leading term of $M(u) - M(\varphi)$ as $\langle L_1 h_1, h_1 \rangle + \langle L_2 h_2, h_2 \rangle$ where L_i are second-order selfadjoint differential operators in $L^2[0,T]$ with potentials depending on r and satisfying $L_1 r' = L_2 r = 0$. The proof of orbital stability requires that zero is the second eigenvalue of L_1 and the first one of L_2.

Equation (14) has the following conservation laws

$$Q(u) = i\int_0^T \overline{u}_x u dx, \quad P(u) = \int_0^T |u|^2 dx, \quad E(u) = \int_0^T (|u_x|^2 - \frac{2|u|^3}{3}) dx.$$

Let us consider the functional

$$M(u) = E(u) + (\omega + v^2)P(u) - 2vQ(u).$$

Next we introduce the pseudometric

$$d(u,\varphi)=\inf_{(\eta,\xi)\in[0,2\pi]\times[0,T]}||u(x,t)-e^{i\eta}\varphi(x-\xi,t)||_1. \tag{16}$$

For a fixed $q>0$, we denote

$$d_q^2(u,\varphi)=\inf_{(\eta,\xi)\in[0,2\pi]\times[0,T]}(||u_x(x,t)-e^{i\eta}\varphi_x(x-\xi,t)||^2+q||u(x,t)-e^{i\eta}\varphi(x-\xi,t)||^2). \tag{17}$$

Theorem 2.10. *For each $\varepsilon>0$ there exists $\delta>0$ such that if $u(x,t)$ is a solution of* (14) *and $d(u,\varphi)_{|t=0}<\delta$, then $d(u,\varphi)<\varepsilon\ \forall t\in[0,\infty)$.*

The crucial step in the proof will be to verify the following statement.

Proposition 2.1. *There exist positive constants Λ,q,δ_0 such that if u is a periodic solution of* (14)*, $u(x,t)=u(x+T,t)$, $P(u)=P(\varphi)$ and $d_q(u,\varphi)<\delta_0$, then*

$$M(u)-M(\varphi)\geq\Lambda d_q^2(u,\varphi). \tag{18}$$

6. **S. Hakkaev, M. Stanislavova, A. Stefanov**, Spectral stability for subsonic traveling pulses of the Boussinesq 'abc' system, **SIAM Journal on Applied Dynamical Systems**, 12(2)(2013), 878-898 (Q1)

In this work, we are concerned with the Boussinesq system

$$\begin{aligned}&\eta_t+u_x+(\eta u)x+au_{xxx}-b\eta_{xxt}=0,\\&u_t+\eta_x+uu_x+c\eta_{xxx}-du_{xxt}=0.\end{aligned} \tag{19}$$

Since it is derived from the Euler equation by ignoring the effects of the dissipation, one generally expects such systems to exhibit a Hamiltonian structure. This is however not generally the case, unless one imposes some further restrictions on the parameters. Indeed, if b = d, one can easily check that

$$H(\eta,u)=-c\eta_x^2-au_x^2+\eta^2+(1+\eta)u^2dx. \tag{20}$$

Furthermore, $H(\eta,u)$ is positive definite only if $a,c<0$. From this point of view, it looks natural to consider the case $b=d$ and $a,c<0$. The solutions of interest are traveling waves, that is, in the form $\eta(x,t)=\varphi(x-wt), u(x,t)=\psi(x-wt)$. There are the (pair of) exact traveling wave solutions

$$\varphi=\eta_0 sech^2(\lambda x),\quad \psi=B(\eta_0)\eta_0 sech^2(\lambda x). \tag{21}$$

In this paper, we will concentrate on spectral stability. There is also (the closely related and almost equivalent) notion of linear stability, which we also mention below. In order to introduce the object of our study, as well as motivate its relevance, let us perform a linearization of the nonlinear system (19). Using the ansatz

$$\begin{aligned} \eta &= \varphi(x - wt) + v(t, x - wt), \\ u &= \psi(x - wt) + z(t, x - wt), \end{aligned} \tag{22}$$

and ignoring all quadratic terms leads to the following linearized problem:

$$(1 - v\partial_x^2) \begin{pmatrix} v \\ z \end{pmatrix}_t = -\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 + c\partial_x^2 & bw\partial_x^2 + \psi - w \\ bw\partial_x^2 + \psi - w & 1 + a\partial_x^2 + \varphi \end{pmatrix} \tag{23}$$

which is of the form

$$u_t = JLu. \tag{24}$$

Definition 2. We say that the problem (24) is unstable if there are $\mathbf{f} \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ and $\lambda : \Re\lambda > 0$ so that

$$JL\mathbf{f} = \lambda\mathbf{f}. \tag{25}$$

Otherwise, the problem (24) is stable. That is, stability is equivalent to the absence of solutions of (25) with $\lambda : \Re\lambda > 0$.

For the case $a = c = -b, b > 0$, we have the following theorem.

Theorem 2.11. *Let $a = c = -b, b > 0$. Then, the traveling wave solutions of the abc system*

$$\left(\eta_0 sech^2 \left(\frac{x - wt}{2\sqrt{b}} \right), \pm\eta_0 \sqrt{\frac{3}{\eta_0 + 3}} sech^2 \left(\frac{x - wt}{2\sqrt{b}} \right) \right) \tag{26}$$

with speed $w = \pm\frac{3+2\eta_0}{\sqrt{3(3+\eta_0)}}$ are stable for $\eta_0 : \eta_0 \in (-\frac{9}{4}, 0)$. Equivalently, all waves in (26) are stable for all subsonic speeds $|w| < 1$.

In the remaining case, we assume only $a = c < 0, b = d > 0$, but we observe that in this case Theorem 2.11 requires that $\eta_0 = -3/2, w = 0$, that is, that the traveling waves become standing waves.

Theorem 2.12. *Let $a = c < 0, b = d > 0$. Then, the standing wave solutions of the Boussinesq system*

$$\varphi(x) = \frac{3}{2} sech^2 \left(\frac{x}{2\sqrt{-a}} \right), \quad \psi(x) = \pm \frac{3}{\sqrt{2}} sech^2 \left(\frac{x}{2\sqrt{-a}} \right) \tag{27}$$

are spectrally stable if and only if

$$\langle (a\partial_x^2 + 1 - \varphi)^{-1}(\varphi - b\varphi''), (\varphi - b\varphi'') \rangle \le 8\sqrt{-a} \left(\frac{9}{2} + \frac{12}{5} \frac{b}{|a|} - \frac{3}{10} \frac{b^2}{a^2} \right). \tag{28}$$

Furthermore, there exists an absolute constant γ so that the condition (28) is satisfied (and hence the waves are stable) if and only if

$$0 < \frac{b}{|a|} < \gamma \sim 8.42083.$$

That is, the waves are unstable in the complementary region $\frac{b}{|a|} > \gamma$. The numerical value of γ is obtained by means of numerical simulations.

7. **S. Hakkaev, M. Stanislavova, A. Stefanov**, Linear stability analysis for periodic traveling waves of the Boussinesq equation and the KGZ system, **Proceedings of the Royal Society of Edinburgh : Section A Mathematics**, 144(03) (2014), 455-489 (Q1)

The question of the linear stability of spatially periodic waves for the Boussinesq equation (in the cases p = 2, 3)

$$u_{tt} + u_{xxxx} - u_{xx} + (f(u))_{xx} = 0, \tag{29}$$

where $f(u)$ is, for the most part, $f(u) = u^p, p > 1$ and the Klein Џ Gordon Џ Zakharov (KGZ)system

$$\begin{array}{l} u_{tt} - uxx + u + un = 0, \\ n_{tt} - n_{xx} = \frac{1}{2}(|u|2)_{xx} \end{array} \tag{30}$$

is considered.

For the Boussinesq equation the question of linear stability of equations in the form

$$z_{tt} + 2wz_{tx} + Hz = 0, \tag{31}$$

or what is equivalent (at least in this case) to the solvability of

$$\lambda^2\psi + 2w\lambda\psi' + H\psi = 0, \tag{32}$$

where $H = -\partial_x L \partial_x$, $L = -partial_x^2 + (1-c^2) - f'(\varphi)$. Here, L is the ubiquitous standard second-order Schrodinger operator, which appears in the linearization of the generalized Korteweg-de Vries equation around its travelling wave solution φ_c. This observation is crucial for the spectral properties of the operator H, as the properties of L are generally well known, at least for the cases under consideration, $p = 2, 3$.

For the KGZ system the question of linear stability of equations in the form

$$\Phi_{tt} - 2c\Phi_{tx} + H\Phi = 0, \tag{33}$$

where $\Phi = \begin{pmatrix} v \\ z \end{pmatrix}$ and

$$H = \begin{pmatrix} H_1 & A \\ A^* & H_2 \end{pmatrix}$$

where

$$H_1 = -(1-c2)\partial_x^2 + 1 - \frac{\varphi^2}{2w}, \quad H_2 = -w\partial_x^2$$
$$Az = \varphi z_x, \quad A^* = -(\varphi z)_x$$

For a wide class of solutions, we completely and explicitly characterize their linear stability (instability) when the perturbations are taken with the same period T.

8. **A. Demirkaya, S. Hakkaev, M. Stanislavova, A. Stefanov**, On the spectral stability of periodic waves of the Klein-Gordon equation, **Differential and Integral Equations**, 28(5,6)(2015), 431-454 (Q2)

The object of study is the Klein-Gordon equation in 1 + 1 dimensions, with integer power non-linearities

$$u_{tt} - u_{xx} + u + |u|^{p-1}u = 0.$$

In particular, of inter- est is the spectral stability/instability (with respect to perturbations of the same period) of traveling-standing periodic solitons

$$u(t,x) = e^{wt}e^{iq(x-ct)}\varphi_{w,c}(x-ct),$$

which are of cnoidal (p = 2), dnoidal (p = 3) or more general type (p = 5). The cor- responding linearized problem for this two-parameter family of solutions fits the general abstract framework of spectral stability for second order Hamiltonian systems

$$v_{tt} + \mathcal{J}v_t + \mathcal{H}v = 0,$$

with associated eigenvalue problem

$$\lambda^2\psi + \lambda\mathcal{J}\psi + \mathcal{H}\psi = 0. \tag{34}$$

We say that the quadratic pencil given by the couple $(\mathcal{J}, \mathcal{H})$ is spectrally unstable, if there exists a T periodic function $\psi \in D(\mathcal{H})$ and $\lambda, \Re\lambda > 0$, so that

$$\lambda^2\psi + \lambda\mathcal{J}\psi + \mathcal{H}\psi = 0.$$

Otherwise, we say that the quadratic pencil $(\mathcal{J}, \mathcal{H})$ is spectrally stable

Our approach is based on the following index counting formula

$$k_r + k_c + k_- = n(\mathcal{H}) - n((I - \mathcal{J}\mathcal{H}^{-1}\mathcal{J})_{Ker(\mathcal{H})}))$$

where k_r is the number of positive solutions of (34), k_c is the number of solutions with positive real part, whereas k_- is the total negative Krein index for the quadratic pencil.

9. **A. Demirkaya, S. Hakkaev**, On the spectral stability of periodic waves for the coupled Schrodinger equations, **Physics Letters A**, 379 (2015), 2908-2914 (Q2)

In this present work we consider the periodic standing wave solutions for the coupled nonlinear Schrödinger equations

$$\begin{aligned} &iu_t + u_{xx} + (\beta|u|^4 + 2\sigma|u|^2|v|^2 + \sigma|v|^4)u = 0 \\ &iv_t + v_{xx} + (\sigma|u|^4 + 2\sigma|u|^2|v|^2 + \gamma|v|^4)v = 0, \end{aligned} \tag{35}$$

where u and v are complex valued functions and β, σ, γ are real parameters. Our purpose is to study the existence and the stability of the periodic standing waves for system (35) of the form $u(x,t) = e^{iwt}\varphi(x)$ and $v(x,t) = e^{iwt}\psi(x)$. Here, we are interested in the spectral stability of the periodic standing waves with respect to perturbations that are periodic with the same period as the corresponding wave solutions.

Linearized system is of the form

$$\frac{d}{dt}\begin{pmatrix} U \\ W \end{pmatrix} = \mathcal{J}\mathcal{L}\begin{pmatrix} U \\ W \end{pmatrix}, \tag{36}$$

where operators $\mathcal{J}$ anti-symmetric and $\mathcal{L}$ is self-adjoint. The stationary solution Φ is spectrally unstable if there exists at least one eigenvalue λ of the operator $\mathcal{J}\mathcal{L}$ with positive real part. Our approach is based on the index counting theory. Let $\mathcal{L}\psi_i = 0,\ \ \mathcal{J}\mathcal{L}\Psi_i = \psi_i$ and U is a matrix with elements $U_{ij} = \langle \mathcal{L}^{-1}\mathcal{J}\psi_i, \mathcal{J}\psi_j \rangle$ For a self-adjoint operator H, define the number of the negative eigenvalues

$$n(H) = \#\{\lambda \in (-\infty, 0) \cap \sigma(H)$$

We have the following formula

$$k_r + 2k_c + 2k_- = n(\mathcal{L}) - n(U), \tag{37}$$

where k_r is the number of positive solutions λ, k_c is the number of solutions λ of with nonzero real and imaginary parts, whereas k_- the number of pairs of purely imaginary eigenvalues with negative Krein signature.

The required spectral properties have been achieved by using Floquet theory and some numerical computations. For computational reasons, the discrete variant of the system (35) has been studied and the numerical standing wave solution to the system (35) has been identified via a Newton-type fixed point method.

10. **S. Hakkaev. M. Stanislavova, A. Stefanov**, Periodic travelling waves of the regularized short pulse and Ostrovsky equations: existence and stability, **SIAM Journal on Mathematical Analysis**, 49(1)(2017), 674-698 (Q1)

We construct various periodic travelling wave solutions of the Ostrovsky/Hunter-Saxton/short pulse equation

$$(u_t + (f(u))_x)_x = u. \tag{38}$$

and its KdV regularized version

$$u_t + \beta u_{xxx} + (f(u))_x + \epsilon\partial_x^{-1}u = 0. \tag{39}$$

First state and prove a general theorem for the existence of periodic travelling waves for (39) for small $\epsilon : |\epsilon| << 1$. It is important to note that for the existence of these waves, an important spectral condition involving the linearized operator

$$L_+ := -\beta\partial_x^2 + c - f'(\varphi_0)$$

Theorem 2.13. *Let the non-linearity f be so that, $f \in C^2(\mathbb{R}), f(0) = 0$ and the following holds There is an even, mean zero and smooth solution $\varphi_0 \in L^2$ of (39) with $\epsilon = 0$. $Ker[L_+] \subset L^2_{odd}$. That is, the kernel of L_+ is spanned by odd functions and $\langle L_+^{-1}[1], 1\rangle \neq 0$.*

Then, there exists $\epsilon_0 > 0$, so that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$, there is a function $\varphi_\epsilon \in H^2_{0,even}$, which is a solution to the equation (39).

Next, we consider the stability of periodic traveling waves constructed in Theorem 2.13

Theorem 2.14. *Assume that the nonlinearity f, the even solution φ of 39 and the operator $L_+ = -\beta\partial_x^2 + c - f'(\varphi)$ satisfy the assumptions of Theorem 2.13. Then, there exists $\epsilon_0 > 0$, so that travelling waves φ_ϵ exist for all $\epsilon \in (-\epsilon_0, \epsilon_0)$.The functions φ_ϵ are even.*

Furthermore, assume that L_+ has a simple and single negative e-value and a simple eigenvalue at zero (with kernel spanned by φ'). Then, under the assumption

$$\langle L_+^{-1}[\varphi], \varphi\rangle < 0 \quad (40)$$

we can conclude that all waves φ_ϵ are linearly stable, when perturbed with perturbations with the same period.

For the short pulse model, we construct a family of travelling peakons with corner crests. We show that the peakons are spectrally stable as well.

11. **S. Hakkaev. M. Stanislavova, A. Stefanov**, Spectral stability for classical periodic waves of the Ostrovsky and short pulse models, **Studies in Applied Mathematics**, 139(3)(2017), 405-433 (Q1)

We consider the short pulse models in a symmetric spatial interval, subject to periodic boundary conditions

$$(u_t + (f(u))_x)_x = u. \quad (41)$$

Our main interest in this paper is the stability of explicit and classical traveling waves for the short pulse equation (38) of the form $u(t,x) = \varphi(x-ct)$. The profile equation for φ is

$$((\varphi^{p-1}-c)\varphi_\xi)_\xi = \varphi, \quad -L \leq \xi \leq L. \tag{42}$$

Clearly, (42) is not very nice object to deal with. We perform a change of variables

$$\xi = \Xi(\eta) := \eta - \frac{\Psi(\eta)}{c}, \quad \varphi(\xi) = \Phi(\eta) = \Psi'(\eta). \tag{43}$$

which leads to the profile equation

$$-c^2\Phi'' - c\Phi + \Phi^p = 0. \tag{44}$$

One has to keep in mind however, that the solutions of (44) are equivalent, so long as the transformation (43) is invertible. For $p=2$ and $p=3$ we construct the explicit expression for Φ,

$$\Phi = \Phi_0 + (\Phi_1 - \Phi_0)sn^2(\alpha x, \kappa), \tag{45}$$

and

$$\Phi = \Phi_2 sn(\alpha x, \kappa), \tag{46}$$

respectively. Our next task is to derive the linearized problem for such solutions φ - assuming that they exist and the transformation (43) is invertible in the appropriate interval. To this end, we take the ansatz $u(t,x) = \varphi(x-ct) + v(t, x-ct)$ in (41) and ignore all quadratic terms. We obtain the following linearized equation

$$(v_t + ((\varphi^{p-1}-c)v)_\xi)_\xi = v \quad -L < \xi < L. \tag{47}$$

Next, we turn (47) into an eigenvalue problem, by letting $v(t,\xi) = e^{\lambda t}w(\xi), w \in H^2[-L,L]$. This results in

$$(\lambda w + ((\varphi^{p-1}-c)w)_\xi)_\xi = w \quad -L < \xi < L. \tag{48}$$

Again we perform a change of variables 43) and we get the eigenvalue problem

$$-c^2 Z_{\eta\eta} - cZ + \Phi^{p-1}Z = -\lambda c Z_\eta, Z \in L^2(-M,M). \tag{49}$$

Our main results are

Theorem 2.15. *The waves described in* (46) *are spectrally stable for all wave speeds* $c > 0$.

and

Theorem 2.16. *The waves described in* (47) *are spectrally stable for all wave speeds* $c > 0$.

12. **S. Hakkaev, M. Stanislavova, A. Stefanov**, On the generation of stable Kerr frequency combs in the Lugiato-Lefever model of periodic optical waveguides, **SIAM Journal on Applied Mathematics**, 79(2) (2019), 477-505 (Q2)

In this paper we consider the following equation

$$iu_t + u_{xx} - u + 2|u|^2 u = -i\alpha u - h, t \geq 0, -T \leq x \leq T \tag{50}$$

where u is complex-valued function, while $\alpha > 0$ is the detuning/damping parameter and the normalized pumping strength parameter is $h > 0$. We are interested in time independent solutions, that is frequency/Kerr combs $u(t, x) = \varphi(x)$ and their stability, as solutions of the full time dependent problem (50). These satisfy the time-independent equation

$$-\varphi'' + \varphi - 2|\varphi|^2\varphi = i\alpha\varphi + h, -T \leq x \leq T \tag{51}$$

Our goal in this paper is to explore the existence and the stability properties of the solutions of (51), in the physically relevant regime $0 < h << 1$. Let us first discuss an impossibility result that we alluded to above.

Proposition 2.2. *($h = 0$ does not support stationary solutions) The equation*

$$\varphi'' - \varphi + 2|\varphi|^2\varphi + i\alpha\varphi = 0, \tag{52}$$

<u>*does not have*</u> *non-trivial classical solutions* φ_α.

In the other endpoint case, that is $\alpha = 0$, one looks for spatially periodic, time-independent solutions of (50), $u = \varphi(x)$. For $\varphi(x)$, we have the equation

$$\varphi'' - \varphi + 2\varphi^3 = -h, -T \leq x \leq T \tag{53}$$

It turns out that this problem has good explicit cnoidal solutions, which we now describe. We integrate once the equation (51), to get

$$\varphi'^2 = -\varphi^4 + \varphi^2 - 2h\varphi - c, \tag{54}$$

where c is a constant of integration. Recall that our interest is in the regime $0 < h << 1$. The solution of (54) is given by

$$\varphi(x) = \frac{\zeta_4(\zeta_3 - \zeta_1) + \zeta_1(\zeta_4 - \zeta_3)sn^2(\frac{x}{\sqrt{g}}, \kappa)}{(\zeta_3 - \zeta_1) + (\zeta_4 - \zeta_3)sn^2(\frac{x}{\sqrt{g}}, \kappa)}, \tag{55}$$

Let $u(t,x) = \varphi(x) + v(t,x)$, where v is a complex-valued function. Plugging this in (50) and ignoring the contributions of all terms in the form $O(v^2)$, we obtain

$$-v_{2t} + v_{1xx} - v_1 + 6\varphi^2 v_1 = 0$$
$$v_{1t} + v_{2xx} - v_2 + 2\varphi^2 v_2 = 0$$

This is clearly in the form

$$\mathcal{J}\begin{pmatrix} \mathcal{L}_{+,h} & 0 \\ 0 & \mathcal{L}_{-,h} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix}$$

where $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\begin{aligned} \mathcal{L}_{+,h} &= -\partial_x^2 + 1 - 6\varphi_h^2 \\ \mathcal{L}_{-,h} &= -\partial_x^2 + 1 - 2\varphi_h^2 \end{aligned}$$

and $L_\pm := \mathcal{L}_{\pm,0}$. Introduce $\mathcal{L}_h := \begin{pmatrix} \mathcal{L}_{+,h} & 0 \\ 0 & \mathcal{L}_{-,h} \end{pmatrix}$. Taking the ansatz $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\mathcal{J}\mathcal{L}_h\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{56}$$

Thus, the stability of the wave φ_h is determined from the eigenvalue problem (56). Following the usual notions of (spectral) stability, we say that the wave is spectrally stable, if (56) has no non-trivial solutions (that is $\vec{v} \neq 0$), $(\lambda, \vec{v}) : \vec{v} \in H^2[-T, T]$ with $\Re\lambda > 0$. Our first result is about the instability of $\varphi = \varphi_h$, as solutions for (55).

Proposition 2.3. *Let $\alpha = 0$. Then the waves* (55) *are spectrally unstable solutions of* (50)*, with a single real eigenvalue, for all sufficiently small values of $h : 0 < h << 1$.*

Theorem 2.17. *Let α_0 is such that $0 < \alpha_0 < \frac{\langle 1,\varphi_0\rangle}{\|\varphi_0\|^2}$. Then, there exists $h_0 = h_0(\alpha_0) > 0$, so that for every $h : 0 < h < h_0$ and $\alpha := \alpha_0 h$, there exists a stationary solution $\varphi_\alpha = \varphi_{\alpha,1} + i\varphi_{\alpha,2}$ of (50). Moreover, there is the following Taylor expansions formula for the coefficients*

$$\varphi_{\alpha,1} = (a_0 + \frac{b_0}{2} h D_2^0 + O(h^2))\varphi_0 + h\Psi_1^0 + O_{\{\varphi_0\}^\perp}(h^2) \tag{57}$$

$$\varphi_{\alpha,2} = (b_0 - \frac{a_0}{2} h D_2^0 + O(h^2))\varphi_0 + h\Psi_2^0 + O_{\{\varphi_0\}^\perp}(h^2) \tag{58}$$

where

$$a_0 = \sigma_0 \frac{\|\varphi_0\|^2}{\langle 1,\varphi_0\rangle}, \quad b_0 = \alpha_0 \frac{\|\varphi_0\|^2}{\langle 1,\varphi_0\rangle}, \sigma_0 = \pm\sqrt{\frac{\langle 1,\varphi_0\rangle^2}{\|\varphi_0\|^4} - \alpha_0^2};$$

$$D_2^0 = 8\frac{\langle \varphi_0^2 L_+^{-1}[1], L_-^{-1}[b_0 - \alpha_0\varphi_0]\rangle}{\langle 1,\varphi_0\rangle};$$

$$\Psi_1^0 = a_0^2 L_+^{-1}[1] + b_0 L_-^{-1}[b_0 - \alpha_0\varphi_0];$$

$$\Psi_2^0 = a_0 b_0 L_+^{-1}[1] - a_0 L_-^{-1}[b_0 - \alpha_0\varphi_0].$$

We also have the following theorem regarding the stability of these solutions.

Theorem 2.18. *Let $h, \alpha_0, \varphi_\alpha(h)$ be as in Theorem 2.17. Then, $\varphi_\alpha(h)$ is stable <u>if and only if</u>*

$$\sigma_0 = -\sqrt{\frac{\langle 1,\varphi_0\rangle^2}{\|\varphi_0\|^4} - \alpha_0^2}.$$

In addition, in the stable case, the spectrum of the full linearized operator $\mathcal{J}\mathcal{L}_h$ has two real eigenvalues 0 and -2α, and the rest of the spectrum is on the vertical line $\{\mu : \Re\mu = -\alpha\}$. That is, it admits the description

$$\sigma(\mathcal{J}\mathcal{L}_h) \subset \{0\} \cup \{-2\alpha\} \cup \{\mu : \Re\mu = -\alpha\}.$$

In the unstable case, which occurs for

$$\sigma_0 = \sqrt{\frac{\langle 1,\varphi_0\rangle^2}{\|\varphi_0\|^4} - \alpha_0^2},$$

there is a single real unstable eigenvalue in the form $\mu_0\sqrt{h} + O(h)$, where

$$\mu_0 = \sqrt{\frac{\sigma_0\|\varphi_0\|^2}{-\langle L_+^{-1}\varphi_0, \varphi_0\rangle}} > 0.$$

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(Assoc. Prof. Dr.Sci. Sevdzhan Hakkaev)