SUMMARIES OF THE PUBLICATIONS

of Vejdi Ismailov Hasanov

included in the documentation of the procedure for the academic position "professor" in the field of higher education 4. Natural science, Mathematics and Informatics, professional field 4.5. Mathematics (Computational mathematics)

(the publications hereby are not part of any other publications or research works required for the scientific PhD title and the academic position of Associate Professor)

Textbooks and textbook manuals

- V. Hasanov, Lectures on numerical methods, (electronic textbook) DLC Shumen university, 2014, 139 pages, ISBN 978-954-577-860-5

  The lectures are grouped into eight chapters. The last chapter has an auxiliary function. Each chapter consists of paragraphs that correspond to the curriculum of Numerical methods in Shumen University. The main topics are: Interpolation by Lagrange interpolating polynomial and Newton's divided difference interpolation formula, Approximation of function on normalized spaces (Uniform approximation, Mean-square approximation and the Method of least squares), Newton – Cotes formulas, Numerical methods for solving nonlinear equations, Direct and iterative methods for solving linear systems of equations and finally, Numerical methods for determining the eigenvalues and eigenvectors of matrices. The last chapter is an appendix of the necessary minimum knowledge in mathematical analysis and algebra. There are control-related issues and tasks aiming to examine the adopted theoretical material.

  The cycle of lectures are mainly structured for the purposes of the students of the Shumen University, but it can also be useful for students from other universities studying the same subject as well as for anyone interested in numerical methods.

- V. Hasanov, Linear optimization, UPH "Bishop Konstantin Preslavsky", Shumen, 2019, 192 pages, ISBN 978-619-201-327-1

  The linear optimization is a main undergraduate course in the curriculum of many undergraduate courses in higher education and in some others it is part of broader-scale mathematics courses.

  This textbook addresses mainly the students of Shumen University, though it can also be useful for other students, PhD students and specialists, who are interested at the theory of linear optimization. It should be taken into consideration that the theoretical coverage in this book is broader than the material covered in the lectures of BA degree students. Contrary to the expectations that the theorems should lack proof grounding here the author has included them in order to satisfy even the most inquisitive and pretentious of readers.

  The textbook is separated into paragraphs which are grouped in five chapters with an appendix supplemented. Each paragraph comprising theoretical explanations is backed up by several examples and/or illustrations which could be helpful in the process of studying. Moreover, at the end of each paragraph there are self-test questions and exercises.

- V. Hasanov, Handbook on Numerical analysis with Matlab, (second edition), UPH "Bishop Konstantin Preslavsky", Shumen, 2019, 248 pages, ISBN 978-619-201-310-3

  The content of this handbook is in accordance with the studied material in "Numerical Methods" discipline at Shumen University. The handbook specifically aim at the students of Shumen University, though it can also be useful for other students, PhD students and specialists, as well as anyone interested in the theory of numerical methods.

  The book is separated into paragraphs grouped in six chapters and an appendix titled "Introduction to Matlab". Each paragraph initiated with a theoretical explanation then continues with certain exemplary exercises and self-test drills. Some of the theoretical explanations are shorter than others but each includes well-based material as an introduction to numerical methods it may reveal which could prove sufficient in order to provide a better understanding of the examples and exercise solving. The last paragraph of each chapter contains the integrated in Matlab functions needed to solve the exercises as well as other functions which take part in the implementation of the studied methods.

  The first edition was published in 2006. The second edition has been revised and supplemented with some new paragraphs referring to: Hermite interpolation problem, Interpolation with spline functions, and the Tridiagonal matrix algorithm.

Papers

1. **V.I. Hasanov**, Notes on two perturbation estimates of the extreme solutions to the equations $X \pm A^*X^{-1}A = Q$, *Applied Mathematics and Computation*, 216, (2010), pp.1355-1362, `(Q1, IF=1.536)`

   In this paper two perturbation estimates are presented of the largest positive definite solutions of the matrix equations

$$X + A^*X^{-1}A = Q \tag{1.1}$$

and

$$X - A^*X^{-1}A = Q, \tag{1.2}$$

have been considered, respectively.

The perturbed equations

$$\tilde{X} + \tilde{A}^*\tilde{X}^{-1}\tilde{A} = \tilde{Q} \tag{1.3}$$

and

$$\tilde{X} - \tilde{A}^*\tilde{X}^{-1}\tilde{A} = \tilde{Q} \tag{1.4}$$

have been considered, respectively.

The main results are:

Theorem 1.1. *Let A, Q, $\tilde{A}$, $\tilde{Q} \in \mathbb{C}^{n\times n}$ $(Q > 0, \tilde{Q} > 0)$ be coefficient matrices for matrix equations (1.1) and (1.3), P is a positive definite matrix. We assume that the equation (1.1) has positive definite solution and X_L is the maximal solution. Let $\alpha_+ = \|PX_L^{-1}AP^{-1}\|_2$, $\beta_+ = \|PX_L^{-1}P\|_2$,*

$$\begin{aligned} b_+ &= 1 - \alpha_+^2 + \beta_+ \left\|P^{-1}\Delta QP^{-1}\right\| , \\ c_+ &= \left\|P^{-1}\Delta QP^{-1}\right\| + 2\alpha_+ \left\|P^{-1}\Delta AP^{-1}\right\| + \beta_+ \left\|P^{-1}\Delta AP^{-1}\right\|^2 . \end{aligned}$$

If $\alpha_+ < 1$ *and*

$$2\left\|P^{-1}\Delta A P^{-1}\right\| + \left\|P^{-1}\Delta Q P^{-1}\right\| \le \frac{(1-\alpha_+)^2}{\beta_+}\,, \qquad (1.5)$$

then $D_+ = b_+^2 - 4c_+\beta_+ \ge 0$, *the perturbed matrix equation (1.3) has the maximal positive definite solution* $\tilde{X}_L$ *and*

$$\|\Delta X_L\| \;\le\; \|P\|_2^2 \frac{b_+ - \sqrt{D_+}}{2\beta_+} \;\equiv\; \|P\|_2^2 S_{err}^+\,.$$

Theorem 1.2. *Let* A, $\tilde{A}$, Q, $\tilde{Q} \in \mathbb{C}^{n\times n}$ $(Q > 0,\ \tilde{Q} > 0)$ *be coefficient matrices for matrix equations (1.2) and (1.4),* P *is a positive definite matrix. Let* $\alpha = \|PX_+^{-1}AP^{-1}\|_2$, $\beta = \|PX_+^{-1}P\|_2$, *where* X_+ *is the unique positive definite solution of equation (1.2),*

$$\begin{aligned} b &= 1-\alpha^2+\beta\|P^{-1}\Delta Q P^{-1}\|\,,\\ c &= \|P^{-1}\Delta Q P^{-1}\| + 2\alpha\|P^{-1}\Delta A P^{-1}\| + \beta\|P^{-1}\Delta A P^{-1}\|^2\,. \end{aligned}$$

If $\alpha < 1$ *and*

$$2\left\|P^{-1}\Delta A P^{-1}\right\|\left\|P^{-1}\Delta Q P^{-1}\right\| \le \frac{(1-\alpha)^2}{\beta}\,, \qquad (1.6)$$

then $D = b^2 - 4c\beta \ge 0$ *and*

$$\|\Delta X_+\| \;\le\; \|P\|_2^2\,\frac{b-\sqrt{D}}{2\beta} \;\equiv\; \|P\|_2^2\, S_{err}\,.$$

These two estimates are modifications of the estimates previously obtained for $P = I$ in:

V.I. Hasanov, I.G. Ivanov, On two Perturbation Estimates of the Extreme Solutions to the Matrix Equations $X \pm A^*X^{-1}A = Q$, *Linear Algebra Appl.*, **413** (2006), pp. 81-92.

The conditions $\|X_L^{-1}A\|_2 < 1$ and $\|X_+^{-1}A\|_2 < 1$ in the above paper are not always satisfied. In this paper (see Theorems 1.1 and 1.2) these conditions have been replaced by $\|PX_L^{-1}AP^{-1}\|_2 < 1$ and $\|PX_+^{-1}AP^{-1}\|_2 < 1$, where P is a positive definite matrix.

In case of $\|Q^{-\frac{1}{2}}AQ^{-\frac{1}{2}}\|_2 < \frac{1}{2}$ has been shown that $\|Q^{\frac{1}{2}}X_L^{-1}AQ^{-\frac{1}{2}}\|_2 < 1$ for Eq. (1.1) and $\|Q^{\frac{1}{2}}X_+^{-1}AQ^{-\frac{1}{2}}\|_2 < 1$ for Eq. (1.2), respectively. In this case can be choose $P = Q^{\frac{1}{2}}$. Unfortunately the question: how to choose the matrix P, such that $\left\|PX_L^{-1}AP^{-1}\right\|_2 < 1$ in Theorem 1.1 and $\left\|PX_+^{-1}AP^{-1}\right\|_2 < 1$ in Theorem 1.2 is an open problem in the present paper.

2. G.H. Nedzhibov, **V.I. Hasanov**, Newton-Secant Method for Solving Operator Equations, *Mathematica Balkanica*, **26**, (2012), pp.369-376, `MathSciNet [MR3184868]`

In this paper two iterative methods have been considered for solving the nonlinear equation

$$F(x) = 0, \qquad (2.1)$$

where F is a Fréchet differentiable operator defined on an open subset D of a Banach space X with values in a Banach space Y. The first method is a generalization of the well known Newton – Secant method for nonlinear scalar equations, and the second method is a modification of the first method.

The following methods are considered:

Newton – secant method

$$
\begin{aligned}
&y_k = x_k - F'(x_k)^{-1}F(x_k)\,,\\
&x_{k+1} = x_k - [x_k, y_k]^{-1}F(x_k)\,, \quad \text{for} \quad k \geq 0\,, \quad \text{and} \quad x_0 \in D
\end{aligned}
\tag{2.2}
$$

and *modification of the Newton – secant method*

$$
\begin{aligned}
&y_k = x_k - F'(x_0)^{-1}F(x_k)\,,\\
&x_{k+1} = x_k - [x_k, y_k]^{-1}F(x_k)\,, \quad \text{for} \quad k \geq 0\,, \quad \text{and} \quad x_0 \in D
\end{aligned}
\tag{2.3}
$$

where $F'(x)$ is a Fréchet derivative of F in x, and $[x, y]$ is a divided difference of F in x and y. It follows that $[x, y](x - y) = F(x) - F(y)$.

Under the assumptions: $\alpha \in D$ is a simple zero of the operator F, at which the $F'(\alpha)^{-1}$ exists, and there exist nondecreasing functions

$$a, b, c : [0, +\infty) \to [0, +\infty)$$

such that:

$$
\begin{aligned}
&\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq a(\|x - \alpha\|)\,, \quad \textit{for} \quad \textit{all} \quad x \in D\\
&\|F'(\alpha)^{-1}([x, y] - [x, \alpha])\| \ \leq b(\|y - \alpha\|)\,, \quad \textit{for} \quad \textit{all} \quad x, y \in D\\
&\|F'(\alpha)^{-1}([x, \alpha] - [\alpha, \alpha])\| \leq c(\|x - \alpha\|)\,, \quad \textit{for} \quad \textit{all} \quad x \in D\,,
\end{aligned}
\tag{2.4}
$$

the following theorem with reference to local convergence of the method (2.2) has been proven:

Theorem 2.1. *Let F be a nonlinear Fréchet differentiable operator defined on a convex open subset D of a Banach space X with values in a Banach space Y. Let us assume that $\alpha \in D$ is a simple zero of the operator F, $F'(\alpha)^{-1}$ exists and conditions (2.4) are satisfied. In addition, let us assume that each of the equations*

$$a(r) + b(r) = 1 \quad \textit{and} \quad 2b(r) + c(r) = 1 \tag{2.5}$$

has a minimum positive zeros r_1 and r_2 respectively. Denote $r = min\{r_1, r_2\}$ and

$$\bar{U}(\alpha, r^*) = \{x \in X : \|x - \alpha\| \leq r^*\} \subset D \quad \textit{for} \quad r^* \in (0, r)\,.$$

Then sequence $\{x_k\}$ $(k \geq 0)$ generated by method (2.2) is well defined, remains in $\bar{U}(\alpha, r^*)$ for all $k \geq 0$, converges to α provided that $x_0 \in \bar{U}(\alpha, r^*)$. Moreover, the following error bounds hold for all $k \geq 0$:

$$\|x_{k+1} - \alpha\| \leq \frac{b(\|y_k - \alpha\|)\|x_k - \alpha\|}{1 - b(\|y_k - \alpha\|) - c(\|x_k - \alpha\|)}\,, \tag{2.6}$$

where $y_k = x_k - F'(x_k)^{-1}F(x_k)$.

Under additional assumption of the above theorem: the function $b(x)/x$ is to be bounded on D that follows cubic convergence of the sequence $\{x_k\}$ defined by (2.2).

By analogy to the above theorem, the convergence of the method (2.3) and its quadratic convergence under boundedness of $b(x)/x$ are proven.

3. A.A. Ali, **V.I. Hasanov**, On some sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = I$, *AIP Conference Proceedings*, Volume 1690, 060001, (2015), Proc. of the 41st International Conference on Applications of Mathematics in Engineering and Economics, AMEE 2015, Sozopol, Bulgaria, 8-13 June 2015, `(SJR=0.18)`

   In this paper the matrix equation

$$X + A^*X^{-1}A - B^*X^{-1}B = I \tag{3.1}$$

   has been examined, where A and B are $n \times n$ matrices, I is $n \times n$ identity matrix.

   Here, weaker sufficient conditions for existence a positive definite solution of Eq. (3.1) are proposed in comparison by these of Duan et al. in Linear and Multilinear Algebra, 62(6):839-852, 2014 and of Berzig et al. in Mathematical Sciences, 2012, 6:27, 2012.

   The conditions $A^*A < \frac{1}{8}I$ and $B^*B < \frac{1}{8}I$ by Duan et al. are replaced by $A^*A < \frac{2}{9}I$ and $B^*B < \frac{2}{2}I$.

   Berzig et al. have proved that Eq. (3.1) has a positive definite solution if there are two parameters $\beta > \alpha > 0$ for which the following conditions are satisfied:

$$\begin{array}{ll} (i) & \frac{1}{\alpha}A^*A + \alpha I \le I \le \beta I; \\ (ii) & \beta A^*A - \alpha B^*B \le \alpha\beta(1-\alpha)I; \\ (iii) & \beta B^*B - \alpha A^*A \le \alpha\beta(\beta-1)I; \\ (iv) & A^*A < \frac{\alpha^2}{2}I, B^*B < \frac{\alpha^2}{2}I. \end{array}$$

   In this paper the above conditions are weakened and replaced with

$$\begin{array}{ll} (a) & \beta A^*A - \alpha B^*B \le \alpha\beta(1-\alpha)I; \\ (b) & \beta B^*B - \alpha A^*A \le \alpha\beta(\beta-1)I; \\ (c) & \|A\|^2 + \|B\|^2 < \alpha^2. \end{array}$$

4. **V.I. Hasanov**, Perturbation Theory for Linearly Perturbed Algebraic Riccati Equations, *Numerical Functional Analysis and Optimization*, **35** (12), (2014), pp.1532-1559, `(Q3, IF=0.591)`

   In this paper perturbation estimates of the stabilizing relative to Π solutions of the equations

$$A^*X + XA - XGX + Q + \Pi(X) = 0 \tag{4.1}$$

   and

$$X - A^*X(I+GX)^{-1}XA - Q - \Pi(X) = 0 \tag{4.2}$$

   have been proposed, respectively. Eqs. (4.1) and (4.2) are linearly perturbed equations by the linear positive operator Π of the classical Riccati equations:

$$A^*X + XA - XGX + Q = 0 \tag{4.3}$$

and

$$X - A^*X(I+GX)^{-1}XA - Q = 0, \tag{4.4}$$

respectively.

The perturbed equations:

$$\tilde{A}^*\tilde{X} + \tilde{X}\tilde{A} - \tilde{X}\tilde{G}\tilde{X} + \tilde{Q} + \tilde{\Pi}(\tilde{X}) = 0 \tag{4.5}$$

and

$$\tilde{X} - \tilde{A}^*\tilde{X}(I+\tilde{G}\tilde{X})^{-1}\tilde{X}\tilde{A} - \tilde{Q} - \tilde{\Pi}(\tilde{X}) = 0 \tag{4.6}$$

are considered, where $\tilde{A} = A + \Delta A$, $\tilde{G} = G + \Delta G$, and $\tilde{Q} = Q + \Delta Q$ are perturbed coefficients of Eqs. (4.1) and (4.2) with ΔA, ΔQ, ΔG, and $\tilde{\Pi} = \Pi + \Delta\Pi$. Here $\Delta\Pi$ is a linear operator, such that $\tilde{\Pi}$ is a positive operator.

Following the Sun's technique in SIAM J. Matrix Anal. Appl., 19 (1998), pp. 39–65 for the Riccati equations (4.3) and (4.4) perturbation estimations have been obtained for the investigated equations.

Let

$$\begin{aligned} \mathbf{L}_c W &= \Phi^* W + W\Phi\,, \\ \mathbf{P}\,N &= \mathbf{L}_c^{-1}\,(XN + N^*X)\,, \\ \mathbf{Q}\,M &= \mathbf{L}_c^{-1}\,(XMX) \end{aligned}$$

and $\Delta\Pi(X) = \Delta_1\Pi(X) + \Delta_2\Pi(X)$, where $\Delta_1\Pi$ and $\Delta_2\Pi$ are perturbation in first and second rate of Π respectively.

Let ε_1, ε_2 and d_π be estimates the following norms:

$$\|\mathbf{L}_c^{-1}\Delta_1\Pi(X)\| \le \varepsilon_1, \quad \|\mathbf{L}_c^{-1}\Delta_2\Pi(X)\| \le \varepsilon_2, \quad \text{and} \quad \|\Delta\Pi\| \le d_\pi,$$

and

$$\begin{cases} l = \|\mathbf{L}_c^{-1}\|^{-1}, \quad p = \|\mathbf{P}\|, \quad q = \|\mathbf{Q}\|, \quad \ell_\pi = \|\mathbf{L}_c^{-1}\Pi\|, \\ \epsilon = \dfrac{1}{l}\|\Delta Q\| + p\|\Delta A\| + q\|\Delta G\|, \quad \varepsilon = \epsilon + \varepsilon_1 + \varepsilon_2, \\ \delta = \|\Delta A\|_2 + \|\Delta G\|_2\|X\|_2, \quad \theta = \delta + \dfrac{d_\pi}{2}, \quad \hat{g} = \|G\|_2 + \|\Delta G\|_2\,. \end{cases} \tag{4.7}$$

One of the main results in the considered paper is:

Theorem 4.1. *Assume that X is a c-stabilizing relative to Π solution to LPCARE (4.1) and for the defined value in (4.7) the following condition is fulfilled:*

$$\theta + \sqrt{l\hat{g}\varepsilon} < (1-\ell_\pi)\frac{l}{2}\,. \tag{4.8}$$

Moreover, let $\tilde{A} = A + \Delta A$, $\tilde{Q} = Q + \Delta Q$, $\tilde{G} = G + \Delta G \ge 0$ and $\Delta\Pi$ is a linear operator, such that $\tilde{\Pi} = \Pi + \Delta\Pi$ is a positive operator. Then perturbed LPCARE (4.5) with coefficient matrices $\tilde{A}$, $\tilde{G}$ and $\tilde{Q}$, and linear operator $\tilde{\Pi}$ has a c-stabilizing relative to $\tilde{\Pi}$ solution $\tilde{X}$, and

$$\|X - \tilde{X}\| \le \frac{2l\varepsilon}{(1-\ell_\pi)l - 2\theta + \sqrt{[(1-\ell_\pi)l - 2\theta]^2 - 4l\hat{g}\varepsilon}} =: \nu_*\,. \tag{4.9}$$

A similar theorem for Eq. (4.2) is proven.

5. **V. Hasanov**, S. Hakkaev, Newton's method for a nonlinear matrix equation, *Comptes rendus de l Academie bulgare des Sciences*, **68** (8), (2015), pp. 973-982, `(Q4, IF=0.284)`

In this paper the Newton's method have been proposed for computing the maximal positive definite solution X_L of the equation

$$X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q. \tag{5.1}$$

Eq. (5.1) appears the first time in the work of Long et al. Bull. Braz. Math. Soc., 39, (2008) 371–386 in case of $m = 2$ and later in the general case in Appl. Math. Comput., 216, (2010) 3480–3485, where the fixed point iteration and its two modifications are considered.

Let $\mathcal{F}(X) = Q - X - \sum_{i=1}^m A_i^* X^{-1} A_i$ and $\mathcal{F}'_X(H) = -H + \sum_{i=1}^m A_i^* X^{-1} H X^{-1} A_i$ where $\mathcal{F}'_X$ is the Fréchet derivative of $\mathcal{F}$ at X.

The Newton's method: $X_{k+1} = X_k - (\mathcal{F}'_{X_k})^{-1}\mathcal{F}(X_k)$ for the equation $F(X) = 0$ can be written

$$X_k - \mathcal{L}_k(X_k) = Q - 2\sum_{i=1}^{m} L_{ik}^* A_i, \quad k = 1, 2 \ldots, \tag{5.2}$$

where $\mathcal{L}_k(X_k) = \sum_{i=1}^m L_{ik}^* X_k L_{ik}$ and $L_{ik} = X_{k-1}^{-1} A_i$.

In the next theorem global convergence of the Newton's method and a property of the maximal solution X_L $\left(\rho(\sum_{i=1}^m (X_L^{-1}A_i)^T \otimes (X_L^{-1}A_i)^*) \leq 1\right)$ are proven. Here $A \otimes B = (a_{ij}B)$ is the Kronecher product of the matrices A and B.

Theorem 5.1. *Let $\hat{X} > 0$ is a solution of $\mathcal{F}(X) \geq 0$ and X_0 such that the matrix $\sum_{i=1}^m (X_0^{-1}A_i)^T \otimes (X_0^{-1}A_i)^*$ is d-stable. Then the Newton sequence $\{X_k\}$ in (5.2) is well defined and satisfies the following:*

- *(a)* $X_k \geq X_{k+1}, \quad X_k \geq \hat{X}, \quad \mathcal{F}(X_k) \leq 0, \quad k \geq 1;$
- *(b)* $\rho(\sum_{i=1}^m (X_k^{-1}A_i)^T \otimes (X_k^{-1}A_i)^*) < 1, \quad k \geq 0;$
- *(c)* $\lim_{k\to\infty} X_k = X_L$ *is the largest solution of (1.1);*
- *(d)* $\rho\big(\sum_{i=1}^m (X_L^{-1}A_i)^T \otimes (X_L^{-1}A_i)^*\big) \leq 1.$

Moreover, it is proven that, if Eq. (5.1) has a positive definite solution, then $\rho\big(\sum_{i=1}^m (Q^{-1}A_i)^T \otimes (Q^{-1}A_i)^*\big) < 1$. Thus, for initial value can be chosen $X_0 = Q$.

6. **V.I. Hasanov**, S.A. Hakkaev, Convergence analysis of some iterative methods for a nonlinear matrix equation, *Computers & Mathematics with Applications*, **72** (4), (2016), pp.1164-1176, `(Q1, IF=1.531)`

In this paper the convergence rate has been examined considering of the following iterative methods for obtaining the maximal solution of Eq. (5.1):

– *Basic fixed point iteration (BFPI)*

$$\begin{cases} X_0 = Q, \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i^* X_k^{-1} A_i, \quad k = 0, 1, \dots; \end{cases} \tag{6.1}$$

– *First inverse free variant of BFPI (FIFV-BFPI)*

$$\begin{cases} X_0 = Q, \quad 0 < Y_0 \leq Q^{-1}, \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i^* Y_k A_i, \\ Y_{k+1} = Y_k(2I - X_k Y_k), \qquad k = 0, 1, \dots; \end{cases} \tag{6.2}$$

– *Second inverse free variant of BFPI (SIFV-BFPI)*

$$\begin{cases} 0 < Y_0 \leq Q^{-1}, \quad X_0 = Q - \sum_{i=1}^{m} A_i^* Y_0 A_i, \\ Y_{k+1} = Y_k(2I - X_k Y_k), \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i^* Y_{k+1} A_i, \qquad k = 0, 1, \dots. \end{cases} \tag{6.3}$$

The main results are presented with respect to the methods considered.

Theorem 6.1. *If Eq. (1.1) has a positive definite solution, then for the BFPI (6.1) we have*

$$\|X_{k+1} - X_L\| \leq \sum_{i=1}^{m} \|X_L^{-1} A_i\|^2 \, \|X_k - X_L\|, \tag{6.4}$$

for all $k \geq 0$.

Moreover, for BFPI (6.1)

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_L\|} \leq \rho\Big(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*\Big) \leq 1$$

is obtained.

Therefore, the convergence of $\{X_k\}$ defined by (6.1) is R-linear whenever $\rho\big(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*\big) < 1$,
and typically sublinear whenever $\rho(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*) = 1$.

Theorem 6.2. *If Eq. (1.1) has a positive definite solution, then for FIFV-BFPI (6.2) and any $\epsilon > 0$, we have*

$$\|Y_{k+1} - X_L^{-1}\| \leq \sum_{i=1}^{m} \|A_i X_L^{-1} + \epsilon\|^2 \, \|Y_{k-1} - X_L^{-1}\| \tag{6.5}$$

and

$$\|X_{k+1} - X_L\| \leq \sum_{i=1}^{m} \|A_i\|^2 \, \|Y_k - X_L^{-1}\| \tag{6.6}$$

for all k large enough. Moreover, if $A_1, A_2, \dots, A_m$ are nonsingular, then

$$\|X_{k+1} - X_L\| \leq \sum_{i=1}^{m} \|X_L^{-1} A_i + \epsilon\|^2 \, \|X_{k-1} - X_L\| \tag{6.7}$$

for all k large enough.

Theorem 6.3. *If Eq. (1.1) has a positive definite solution, for SIFV-BFPI (6.3) and any $\epsilon > 0$, we have*

$$\|Y_{k+1} - X_L^{-1}\| \le \sum_{i=1}^{m} \|A_i X_L^{-1} + \epsilon\|^2 \|Y_k - X_L^{-1}\| \tag{6.8}$$

and

$$\|X_k - X_L\| \le \sum_{i=1}^{m} \|A_i\|^2 \|Y_k - X_L^{-1}\| \tag{6.9}$$

for all k large enough. Moreover, if $A_1, A_2, \dots, A_m$ are nonsingular, then

$$\|X_{k+1} - X_L\| \le \sum_{i=1}^{m} \|X_L^{-1} A_i + \epsilon\|^2 \|X_k - X_L\| \tag{6.10}$$

for all k large enough.

For *SIFV-BFPI* (6.3)

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_L\|} \le \rho\Big(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*\Big) \tag{6.11}$$

is obtained also, i.e. (6.3) has the same convergence rate as the BFPI (6.1).

In addition to the investigated convergence of the above methods, in this paper a modification of the Newton's method (Stein iteration) is proposed:

Let $X_0 = Q$ (X_0 can be chosen by deferent way, see Theorem 6.4). For $k = 1, 2, \dots,$ take $L_{ik} = X_{k-1}^{-1} A_i$, $i = 1, 2, \dots, m$ and solve

$$X_k - L_{jk}^* X_k L_{jk} = Q - \sum_{i=1}^{m} A_i^* L_{ik} - A_j^* L_{jk}, \tag{6.12}$$

where j is fixed index in $\{1, 2, \dots, m\}$, until $\|X_k - X_{k-1}\|_\infty \le tol$. Then $X_L \approx X_k$.

Theorem 6.4. *Let $\hat{X} > 0$ be a solution of $\mathcal{F}(X) \ge 0$, X_0 is a Hermitian matrix for which $\mathcal{F}(X_0) \le 0$ and $\rho(\sum_{i=1}^{m} (X_0^{-1} A_i)^T \otimes (X_0^{-1} A_i)^*) < 1$. Then the iteration (6.12) defines a sequence $\{X_k\}$ with the following properties:*

(a) $X_k \ge X_{k+1}, \quad X_k \ge \hat{X}, \quad \mathcal{F}(X_k) \le 0, \quad k \ge 0$;

(b) $\rho(\sum_{i=1}^{m} (X_k^{-1} A_i)^T \otimes (X_k^{-1} A_i)^*) < 1, \quad k \ge 0$;

(c) $\lim_{k\to\infty} X_k = X_L$ *is the largest solution of (1.1);*

(d) $\rho(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*) \le 1$.

It is proven that, if Eq. (5.1) has a positive definite solution, then $\mathcal{F}(Q) \le 0$. Moreover, when $\rho(\sum_{i=1}^{m} (X_L^{-1} A_i)^T \otimes (X_L^{-1} A_i)^*) < 1$, the R-linear convergence of the Stein iteration (6.12) is proven.

7. **V.I. Hasanov**, A.A. Ali, On convergence of three iterative methods for solving of the matrix equation $X+A^*X^{-1}A+B^*X^{-1}B=Q$, *Computational and Applied Mathematics*, **36** (1), (2017), pp.79-87, `(Q3, IF=0.863)`

   The investigations in this paper precedes the above one, although it was later printed. Here above result for the considered tree methods for Eq. (5.1): BFPI (6.1) and its two inverse free variants FIFV-BFPI (6.2) and SIFV-BFPI (6.3), in case of $m=2$ are obtained.

8. **V.I. Hasanov**, On perturbation estimates for the extreme solution of a matrix equation, *Annals of the Academy of Romanian Scientists: Series on Mathematics and its Applications*, **9** (1), (2017), pp.74-88, `(SJR=0.189)`

   In this paper some perturbation estimates have been obtained for the unique positive definite solution of the equation

$$X-\sum_{i=1}^{m} A_i^*X^{-1}A_i=Q\,.$$

Above equation can be written:

$$X-A^*\widehat{X}^{-1}A=Q\,, \tag{8.1}$$

where $\widehat{X}=I_m\otimes X$ and

$$A=\begin{pmatrix} A_1\\ \vdots \\ A_m\end{pmatrix}.$$

It is proven in literature that, Eq. (8.1) has an unique positive definite solution, which is called extreme.

The perturbed equation

$$\tilde{X}-\tilde{A}^*\widehat{\tilde{X}}^{-1}\tilde{A}=\tilde{Q}\,, \tag{8.2}$$

is considered, where $\tilde{A}$ and $\tilde{Q}$ ($\tilde{Q}>0$) contain small perturbations of A and Q in (8.1), respectively. Let

$$\tilde{A}=\begin{pmatrix} \tilde{A}_1\\ \vdots \\ \tilde{A}_m\end{pmatrix},$$

where $\tilde{A}_i$ contain small perturbations of A_i, $i=1,2,\dots,m$. Let $\tilde{X}_+$ be extreme solution of Eq. (8.2), $\Delta X_+=\tilde{X}_+-X_+$, $\Delta Q=\tilde{Q}-Q$ and $\Delta A=\tilde{A}-A$.

Yin and Fang in J. Appl. Math. Comput., 43, (2013) 199-211 have generalize our estimate from Linear Algebra Appl., 413, (2006) 81-92 for Eq. (8.1) in case of $m=1$ as the condition $\|X_+^{-1}A\|_2<1$ is replaced by $\|A\|_2\|X_+^{-1}\|_2<1$, which is obviously stronger.

Here, although the above condition has been weakened, the technique in the first paper in this list has been applied and the following results have been obtained:

Theorem 8.1. *Let A, Q and $\tilde{A}$, $\tilde{Q}$ with Q, $\tilde{Q}$ positive definite be coefficient matrices for the matrix equations (8.1) and (8.2), respectively, P is a positive definite matrix. Denote $\alpha = \|\widehat{PX_+^{-1}}AP^{-1}\|_2$, $\beta = \|PX_+^{-1}P\|_2$, where X_+ is the extreme solution of Eq. (8.1),*

$$b = 1 - \alpha^2 + \beta\|P^{-1}\Delta QP^{-1}\|_2\,,$$
$$c = \|P^{-1}\Delta QP^{-1}\|_2 + 2\alpha\|\widehat{P^{-1}}\Delta AP^{-1}\|_2 + \beta\|\widehat{P^{-1}}\Delta AP^{-1}\|_2^{\;2}\,.$$

If $\alpha < 1$ and

$$2\|\widehat{P^{-1}}\Delta AP^{-1}\|_2 + \|P^{-1}\Delta QP^{-1}\|_2 \leq \frac{(1-\alpha)^2}{\beta}\,, \tag{8.3}$$

then $D = b^2 - 4c\beta \geq 0$ and

$$\|\Delta X_+\|_2 \;\leq\; \|P\|_2^2\frac{b-\sqrt{D}}{2\beta} \;\equiv\; S_{err}^P\,.$$

The question: how to chose the matrix P such that $\|\widehat{PX_+^{-1}}AP^{-1}\|_2 < 1$, which is appeared in the first paper in this list, here is an open problem, also. Here are some numerical examples are considered for which $P = Q^{1/2} + 2Q^{1/4}$ is proposed, but this value of P does not solve the problem in the general case.

9. **V.I. Hasanov**, D.I. Borisova, Perturbation estimates for the maximal solution of a nonlinear matrix equation, *Annals of the Academy of Romanian Scientists: Series on Mathematics and its Applications*, **9** (1), (2017), pp.28-43, `(SJR=0.189)`

In this paper some perturbation estimations are derived for the maximal positive definite solution of Eq. (5.1):

$$X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q.$$

The perturbed equation

$$\tilde{X} + \sum_{i=1}^{m} \tilde{A_i}^* \tilde{X}^{-1} \tilde{A}_i = \tilde{Q} \tag{9.1}$$

is considered, where the matrix coefficients $\tilde{A}_i := A_i + \Delta A_i$, $i = 1, 2, \ldots, m$ and $\tilde{Q} := Q + \Delta Q$ contain small perturbations.

Duan et al. in Appl. Math. Comput. 218 (2011) 4458-4466 have investigated Eq. (5.1) in case of $Q = I$ and has been obtained perturbation estimate for the maximal solution at perturbation in the matrix coefficients A_i, $i = 1, 2, \ldots, m$.

The first result here is a generalization of the estimate of Duan et al., which is applicable for Eq. (5.1) for general Q with its perturbation:

Theorem 9.1. *Let*

(i) $\|Q^{-1}\|^2 \sum_{i=1}^{m} \|A_i\|^2 < \frac{1}{4}$;

(ii) $\|\Delta Q\| \leq \left[\frac{1}{2} - \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}}\right]\|Q^{-1}\|^{-1}$,

(iii) $\sum_{i=1}^{m}\|\Delta A_i\|^2 + 2\sum_{i=1}^{m}\|A_i\|\|\Delta A_i\| < \left[\frac{1}{4} - \|\tilde{Q}^{-1}\|^2\sum_{i=1}^{m}\|A_i\|^2\right]\|\tilde{Q}^{-1}\|^{-2}$.

Then the equations (9) and (9.1) have maximal solutions X_L and $\tilde{X}_L$, respectively. Moreover,

$$\|\Delta X_L\| \leq \frac{1}{c_1}\left[\|\Delta Q\| + 2\|\tilde{Q}^{-1}\|\sum_{i=1}^{m}\|\Delta A_i\|(2\|A_i\| + \|\Delta A_i\|)\right] =: E_1,$$

where

$$c_1 = 1 - 4\|Q^{-1}\|\|\tilde{Q}^{-1}\|\sum_{i=1}^{m}\|A_i\|^2.$$

The second result is a generalization of the Xu's estimation in Linear Algebra Appl. 336, (2001) 61-70 for Eq. (5.1) in case of $m = 1$. Here the estimate does not depend on the matrix coefficients in the perturbed equation (9.1):

Theorem 9.2. *Let*

(i) $\eta := \frac{1}{2} - \|Q^{-1}\|\Big(\sum_{i=1}^{m}\|A_i\|^2\Big)^{\frac{1}{2}} > 0$,

(ii) $\|\Delta Q\| \leq \eta\|Q^{-1}\|^{-1}$,

(iii) $\sum_{i=1}^{m}\|\Delta A_i\|^2 + 2\sum_{i=1}^{m}\|A_i\|\|\Delta A_i\| < \frac{\eta(2-3\eta)}{4\|Q^{-1}\|^2}$.

Then the equations (9) and (9.1) have maximal solutions X_L and $\tilde{X}_L$, respectively. Moreover

$$\|\Delta X_L\| \leq \frac{1}{c_3}\left[(1-\eta)\|\Delta Q\| + 2\|Q^{-1}\|\sum_{i=1}^{m}\big(2\|A_i\| + \|\Delta A_i\|\big)\|\Delta A_i\|\right] =: E_3,$$

where $c_3 = \eta(3 - 4\eta)$.

10. **V.I. Hasanov**, On a perturbation estimate for the extreme solution of the matrix equation $X - A^*\widehat{X}^{-1}A = Q$, *Innovativity in Modeling and Analytics Journal of Research*, **2**, (2017), pp.1-11.

    In this paper the perturbation estimate which is obtained in the work of the eighth paper in this list has been considered for the extreme solution of the equation $X - A^*\widehat{X}^{-1}A = Q$. In the eighth paper the following question: "How to choose the matrix P such that $\|\widehat{PX_+^{-1}AP^{-1}}\|_2 < 1$" is asked. The same question has been asked in the first paper in case of $m = 1$. In this paper $\big\|\widehat{X_+^{-1/2}AX_+^{-1/2}}\big\|_2 < 1$ is proven, where X_+ is the extreme solution of the considered equation.

    Therefore, the one answer of the above question is $P = X_+^{-1/2}$.

11. **V.I. Hasanov**, On the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = I$, *Linear and Multilinear Algebra*, **66** (9), (2018), pp.1783-1798, `(Q2, IF=0.964)`

In this paper the matrix equation (3.1):

$$X + A^*X^{-1}A - B^*X^{-1}B = I$$

have been investigated.

Some sufficient and necessary conditions are obtained for existence of a positive definite solution of the considered equation, as well as a condition for existence the minimal positive definite solution. Together with the conditions for existence of the positive definite solution, sets of matrices are defined, in which there is a solution.

Theorem 11.1. *Let M be the largest positive definite solution of $X + A^*X^{-1}A = I$. Then Eq. (3.1) has a solution $X_M \in [M, N]$, where N is the largest positive definite solution of $X + A^*X^{-1}A = I + B^*M^{-1}B$. Moreover,*

(i) if $M < X_M$, then $\rho(X_M^{-1}A) \leq 1$,

(ii) if $M < X_M$, and A or B is nonsingular, then $\rho(X_M^{-1}A) < 1$,

(iii) if B is nonsingular, then $M < X_M$.

In the next theorem, the necessary conditions for one positive definite solution of Eq. (3.1) to be less than or greater of I, respectively, are given:

Theorem 11.2. *Let $\hat{X}$ be a Hermitian solution of Eq. (3.1).*

(i) If $\hat{X} \geq I$, then Eq. (3.1) with $Q = I + B^*B$ has the largest positive definite solution X_L and $\hat{X} \leq X_L$. Moreover, if the matrix B is nonsingular, then $\rho(AB^{-1}) \leq 1$.

(ii) If $\hat{X} > I$, then B is nonsingular and $\rho(AB^{-1}) < 1$.

(iii) If $0 < \hat{X} \leq I$ and the matrix A is nonsingular, then $\rho(BA^{-1}) \leq 1$.

(iv) If $0 < \hat{X} < I$, then A is nonsingular and $\rho(BA^{-1}) < 1$.

For the iterative method

$$X_{k+1} = A(I - X_k + B^*X_k^{-1}B)^{-1}A^*, \quad X_0 = \gamma I \tag{11.1}$$

the following theorem is formulated:

Theorem 11.3. *Let A be nonsingular, $0 < A^*A - B^*B \leq \frac{1}{4}I$, $\alpha_1 = \frac{1-\sqrt{1-4\lambda_n(A^*A-B^*B)}}{2}$, $\beta_1 = \frac{1-\sqrt{1-4\lambda_1(A^*A-B^*B)}}{2}$, and $\beta_2 = \frac{1+\sqrt{1-4\lambda_1(A^*A-B^*B)}}{2}$. Then the sequence $\{X_k\}$ defined by (11.1) with*

(i) $\gamma = \gamma_1 \in (0, \alpha_1]$ is monotonically increasing and converges to a positive definite solution $X_{\gamma_1} \in [\gamma_1 I,\ \beta_1 I]$ of Eq. (3.1),

(ii) $\gamma = \gamma_2 \in [\beta_1, \beta_2]$ is monotonically decreasing and converges to a positive definite solution $X_{\gamma_2} \in [\alpha_1 I,\ \gamma_2 I]$ of Eq. (3.1),

(iii) $\gamma \in [\alpha_1, \beta_1]$ and if $\beta_1\|A\|\sqrt{\alpha_1^2 + \|B\|^2} < \alpha_1(\lambda_1(A^*A - B^*B) + \sigma_n^2(B))$, then Eq. (3.1) has a unique solution $\hat{X} \in [\alpha_1, \beta_1]$.

In this paper a second iterative method have been considered, where the matrix sequence $\{Y_k\}$ is generated by the equation

$$Y_k - A^{-*}B^*Y_kBA^{-1} = A^{-*}(I - Y_{k-1}^{-1})A^{-1}, \quad k = 1, 2, \ldots, \tag{11.2}$$

where Y_0 is a solution of

$$Y - A^{-*}B^*YBA^{-1} = A^{-*}A^{-1}. \tag{11.3}$$

Theorem 11.4. *Eq. (3.1) has a positive definite solution $\hat{X} < I$ if and only if A is nonsingular, $\rho(A^{-1}B) < 1$, and there is a Hermitian matrix $\bar{Y} > I$ such that $Y_k > \bar{Y}$ for each element of the sequence $\{Y_k\}_{k=0}^{\infty}$ generated by (11.2)-(11.3).*

From the above mentioned theorem it follows that, if Eq. (3.1) has a positive definite solution $\hat{X} < I$, then it has minimal positive definite solution $X_S = Y_L^{-1} > Y_0^{-1}$, where $Y_L = \lim_{k\to\infty} Y_k$.

12. D.I. Borisova, **V.I. Hasanov**, On some perturbation bounds for a matrix equation from interpolation problems, *Annals of the Academy of Romanian Scientists: Series on Mathematics and its Applications*, **10** (2), (2018), pp.297-313, (SJR=0.354)

In this paper the existing perturbation estimates for the unique positive definite solution of Eq. (8.1):

$$X - A^*\widehat{X}^{-1}A = Q$$

have been compared to numerus numerical examples.

The estimates of the following authors have been considered:

– Sun: Perturbation analysis of the matrix equation $X = Q + A^H(\hat{X} - C)^{-1}A$, Linear Algebra Appl., 362:211-228, 2003;

– Yin and Fang: Perturbation analysis for the positive definite solution of the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^*X^{-1}A_i = Q$, J. Appl. Math. Comput., 43:199-211, 2013;

– Konstantinov et al.: Sensitivity of the matrix equation $A_0 + \sum_{i=1}^{k} \sigma_i A_i^* X^{p_i} A_i = 0$, $\sigma_i = \pm 1$, Appl. Comput. Math., 10:409-427, 2011.

– Hasanov: On perturbation estimates for the extreme solution of a matrix equation Ann. Acad. Rom. Sci. Ser. Math. Appl., 9:74-88, 2017 (the eighth paper in this list) and On a perturbation estimate for the extreme solution of the matrix equation $X - A^*\widehat{X} - 1A = Q$, Innovativity in Modeling and Analytics Journal of Research, 2:1-11, 2017 (the tenth paper in this list);

The results of the experiments have shown that the estimates of Konstantinov et al., the Sun's and the our estimates in many cases have competitive indicators. Moreover, our estimates have simple computational formulas.

Shumen,
2019

Composed by: (Vejdi Hasanov)