SUMMARY OF SCIENTIFIC PUBLICATIONS

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for participation in a competition for the academic position of "associate professor"in the field of higher education 4. Natural Sciences, Mathematics and Informatics and professional field 4.5. Mathematics (Mathematical Analysis) at the Shumen University "Episkop Konstantin Preslavski announced in the State Gazette, issue 63/ 01.08.2025.

(the publications do not repeat those submitted for the acquisition of the educational and scientific degree "doctor")

ARTICLES

The articles submitted in the competition can be summarized in the following groups:

1. Applying the fixed point pair theorem in a partially ordered metric space to find a positive definite solution to a system of matrix equations;
2. Study of tripled fixed points and their application in studying the dynamics of oligopolistic markets with three dominant firms and finding a positive definite solution to a system of matrix equations;
3. Application of the N-tupled fixed point theorem in a partially ordered metric space to find a positive definite solution to a system of matrix equations. Finding a connection between matrix equations or systems and the partial ordering they give rise to in the considered Cartesian products, which allows to expand the class of problems solvable with the theory of N-orc fixed points.

The obtained results are based on several fundamental papers in the field of fixed points for monotone images in partially ordered metric rostrants [A. C. M. Ran, M. C. B. Reurings], pairs of fixed points in partially ordered metric spaces [T. G. Bhaskar, V. Lakshmikantham], application to solving nonlinear matrix equations using Bhaskar-Lakshmikantham results, [M. Berzig, X. F. Duan, B. Samet].

Last but not least, we would like to note that the ideas for the research started from a report by professor Vezhdi Hasanov at MATTEX 2022.

1. Applying the fixed point pair theorem in a partially ordered metric space to find a positive definite solution to a system of matrix equations

A-1 ***A. Ali, C. Dinkova, A. Ilchev, B. Zlatanov, Bhaskar-Lakshmikantham fixed point theorem vs Ran-Reunrings one and some possible generalizations and applications in matrix equations, AIMS Mathematics, 2024, vol. 9(8), pp. 21890-21917. ISSN: 2473-6988. doi: 10.3934/math.20241064,*** **[B-4.1.]**;

The first results on the existence and unique of a pair of fixed points are obtained by D. J. Guo, V. Lakshmikantham, where the underlying space is assumed to be Banach. Later, easier to application results regarding a pair of fixed points in a complete metric space with introduced partial ordering were obtained by T. G. Bhaskar, V.

Lakshmikantham, which is a starting point for numerous generalizations. In this paper, we show that there is a close connection between the type of monotonicity of the maps under study and the definition of partial ordering in the proposed idea by T. G. Bhaskar, V. Lakshmikantham of studying coupled fixed points for maps with the mixed monotone property in partially ordered metric spaces.

The problem of identifying coupled fixed points can be seen as a problem of solving a system of symmetric equations,

$$\left| \begin{array}{l} x = F(x,y) \\ y = F(y,x). \end{array} \right.$$

Zlatanov suggested to modify the concept of coupled fixed points so that to solve arbitrary systems of two equations using coupled fixed point results. This allows one to find coupled fixed points (x, y) such that $x \neq y$. The main application of the results by Zlatanov is in altering the problem of finding coupled best proximity points into a task of determining the distance between sets and thus finding the exact solution instead of an approximate one, as proposed by some computer algebra systems. We prove the existence and uniqueness of a coupled fixed point in the case when the maps F and G are continuous. An estimate of the best approximation error is given.

M. Berzig, X. F. Duan, B. Samet investigate the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = Q$, where A and B are arbitrary square $N \times N$ matrices and Q is a positive definite $N \times N$ matrix. Using the main result of T. G. Bhaskar, V. Lakshmikantham, they reduce the matrix equation to a system of two symmetric matrix equations

$$\left| \begin{array}{l} X = Q - A^*X^{-1}A + B^*Y^{-1}B \\ Y = Q - A^*Y^{-1}A + B^*X^{-1}B. \end{array} \right.$$

Following Bhaskar & Lakshmikantham and the observations by Zlatanov, we extend the notion of mixed monotone maps so that we can enlarge, in a unified manner, the class of problems that can be investigated, and the known results will be a particular case of the new ones. We present adequate criteria to guarantee the uniqueness and existence of solutions for the system of matrix equations

$$\left| \begin{array}{l} X = Q - A^*X^{-1}A + B^*Y^{-1}B \\ Y = R - C^*X^{-1}C + D^*Y^{-1}D. \end{array} \right.$$

We provide a generalization of the results by A. C. M. Ran, M. C. B. Reurings of the existence and uniqueness of fixed points in partially ordered metric spaces for a monotone map by removing the assumption on the continuity of the map. We alter the notion of coupled fixed points so that the solution (X, Y) does not need to satisfy $X = Y$. We apply the major results in the investigation of coupled fixed points for ordered pairs of two maps that meet various monotone features, including a mixed monotone property or a total monotone property, which generalizes the known investigations on the topic.

In article [A-1] we construct a generalization of a pair of fixed points. We show that there is a close connection between the type of monotonicity of the maps under study and the definition of a partial order for studying coupled fixed points for maps in partially ordered metric spaces. Instead of considering maps that are only of the class of those with the mixed monotonic property, we show that it is also possible for them to have the same type of monotonicity in each of their two variables. In this case, the partial ordering in the Cartesian product is changed, which allows for the ordered pair of maps again $T(x, y) = (F(x, y), G(x, y))$ to be monotonic, according to the newly introduced regulation, and the results of the A. C. M. Ran, M. C. B. Reurings and the generalization we have made. Considering an ordered pair of maps (F, G) provides the opportunity to solve not only symmetric systems of matrix equations. In addition, the obtained sufficient conditions coincide with the known sufficient conditions from M. Berzig, X. F. Duan, B. Samet, which confirms that the new results obtained generalize the previously known ones. The results we obtained expand the classes of systems of matrix equations that we can solve.We generalize the main result of A. C. M. Ran, M. C. B. Reurings on the existence and uniqueness of fixed points, showing that even without continuity of the mapes, the same conclusions are valid.

A-2 ***A. Ali, C. Dinkova, H. Kulina, A. Ilchev, B. Zlatanov, Positive Definite Solution of System of Matrix Equations with X-k and Y-l via Coupled Fixed Point Theorem in Partially Ordered Spaces, Axioms, 14(2), Article number 123, (2025), doi: 10.3390/axioms14020123, ISSN: 2075-1680,*** **[Γ-7.1.]**;

In this work, we consider the equation $X = Q + pA^*X^kA + qB^*X^lB$ for $p, q \in \{\pm 1\}$ and $k, l \in [-1, 1]\backslash\{0\}$, thus, we will consider two monotone functions F_1 and F_2 instead of one F and different signs in the summation. We find sufficient conditions for the existence and unique of solutions for the studied classes of matrix equations. At first glance, the proposed results are similar to those of article [A-1]. We would like to comment that when we can choose the power of the matrices $k, l \in (-1, 1)\backslash\{0\}$, the generalized result obtained in [A-1] is better seen. It turns out that the idea proposed by T. G. Bhaskar, V. Lakshmikantham and D. J. Guo, V. Lakshmikantham and applied in work by M. Berzig, X. F. Duan, B. Samet requires a specific definition of the partial order in the Cartesian product $X \times X$.

We demonstrate that depending on the sign of the power k and the sign $p \in \{\pm 1\}$, with which the addends pA^*X^kA and pC^*X^kC, C participate, they naturally give rise to the partial order in the Cartesian product $X \times X$. The same is true for the addends qB^*Y^lB and qD^*Y^lD, where $q \in \{\pm 1\}$. That is to say, if we consider the system of matrix equations

$$\left| \begin{array}{l} X = Q + pA^*X^kA + qB^*Y^lB \\ Y = R + pC^*X^kC + qD^*Y^lD, \end{array} \right.$$

for $k, l \in (-1, 1)\backslash\{\pm 1\}$ and $p, q \in \{\pm 1\}$, then we can introduce a partial order in $(X \times X, \preccurlyeq)$ as follows::

1. If $sign(k) * p * sign(l) * q > 0$, then $(x, y) \preccurlyeq (u, v)$, if $x \preccurlyeq u$ and $y \preccurlyeq v$.
2. If $sign(k) * p > 0$ and $sign(l) * q < 0$, then $(x, y) \preccurlyeq (u, v)$, if $x \preccurlyeq u$ and $y \succcurlyeq v$.
3. If $sign(k) * p < 0$ and $sign(l) * q < 0$, then $(x, y) \preccurlyeq (u, v)$, if $x \succcurlyeq u$ and $y \succcurlyeq v$.

Received conditions have been obtained for the existence and unique of solutions to systems of two matrix equations when the unknown matrices are raised to a power $k \in [-1,1]\backslash\{0\}$, which is a real number in the specified interval, not an integer between ± 1. We develop the idea from an article [A-1], that the mixed monotone property for the studied maps is not essential. We show that the sign of p in front of the matrix additive pA^*X^kA and the sign of the power k to which the unknown matrix is raised determine which partial ordering to choose in the Cartesian product. This results in ordered pairs of maps that can be with the mixed monotone property or with the same type of monotone in each of their variables.

2. Study of tripled fixed points and their application in studying the dynamics of oligopolistic markets with three dominant firms and finding a positive definite solution to a system of matrix equations;

A-3 ***A. Ali, A. Ilchev, V. Ivanova, H. Kulina, P. Yaneva, B. Zlatanov, Modeling the Tripodal Mobile Market Using Response Functions Instead of Payoff Maximization. Mathematics, 13(1), Article number 171,(2025). ISSN:2227-7390 (online), https://doi.org/10.3390/math13010171;*** **[B-4.2.]**;

In this article, we demonstrate the application of tripled fixed points in the study of oligopolistic markets. Modern markets are characterized by the presence of relatively few participants who control a substantial share of the market. In recent years, a considerable amount of research has been devoted to the analysis of oligopolistic structures. Most studies employ the classical Cournot model (Cournot, A.A.) for payoff maximization. As is well known, the conventional profit-maximization model has certain limitations. These can be formulated as additional requirements for the differentiability of the payoff function, concavity conditions, or second-order conditions implying twice differentiability. Even if these restrictions do not significantly limit the analyzed models, the assumption of rational behavior of market participants has certain drawbacks.

One such drawback is the potential inaccuracy of the price function in its modeling, which may lead to a model that deviates from actual market behavior and produces solutions inconsistent with rational conduct. Another drawback is that participants may be constrained by long-term contracts or technological limitations that prevent them from making substantial adjustments to production volumes, leading to non-rational behavior.

Nevertheless, in an oligopolistic market, all participants have a clear understanding of what changes in production quantities they can make. Knowing both their own market outcomes and those of their competitors—quantities, prices, marketing strategies - they can adjust their policies according to the current market situation. This gives rise to the so-called reaction functions of each participant. These functions rely on prevailing market conditions to determine the modifications each participant should make.

The idea of investigating market equilibrium and its stability through the use of paired fixed points or pairs of points of best proximity was introduced by Dzhabarova, Kabaivanov, Ruseva, and Zlatanov and later developed in a series of papers. A number of studies have shown that in an oligopolistic market with rational participants and differentiable profit functions, finding the equilibrium through profit maximization and finding it via reaction functions lead to identical results.

In [A-3], we examine the U.S. mobile services market, which represents a typical oligopoly with three dominant firms. The reaction functions are modeled over time using the least squares method; real data and their moving-average–smoothed versions are analyzed. Different models—linear and nonlinear—are constructed. Statistical analyses are performed on all model-predicted data, demonstrating the adequacy of the obtained models.

We would like to emphasize the well-known fact that modeling discrete data by continuous functions allows for an infinite number of statistically reliable models. This is why we selected several different classes of functions in our modeling and analyzed which of them best predict the real data. We believe that, regardless of consumer behavior or incomplete knowledge of their utility functions, market participants react to every change.

Combining all the above considerations, we can conclude that the proposed approach to finding market equilibrium through reaction functions has the advantage of being applicable even when only minimal information is available.

A-4 ***A. Ali, C. Dinkova, H. Kulina, A. Ilchev, B. Zlatanov, Tripled Fixed Points, Obtained by Ran-Reunrings Theorem for Monotone Maps in Partially Ordered Metric Spaces, Mathematics 2025, 13, 758. ISSN:2227-7390 (online), https://doi.org/10.3390/math13050758,*** **[Г-7.2.]**;

In this paper we establish criteria for guaranteeing the uniqueness and existence of solutions for the matrix equation system:

$$\left| \begin{array}{l} X = Q_1 + p_1 A_1^* X^{-k_1} A_1 + p_2 B_1^* Y^{-k_2} B_1 + p_3 C_1^* Z^{-k_3} C_1 \\ Y = Q_2 + p_1 A_2^* X^{-k_1} A_2 + p_2 B_2^* Y^{-k_2} B_2 + p_3 C_2^* Z^{-k_3} C_2 \\ Z = Q_3 + p_1 A_3^* X^{-k_1} A_3 + p_2 B_3^* Y^{-k_2} B_3 + p_3 C_3^* Z^{-k_3} C_3, \end{array} \right.$$

where $p_i \in \{-1, 1\}$ and $k_i \in (0, 1]$ for $i = 1, 2, 3$. Our work generalizes the results of A. C. M. Ran, M. C. B. Reurings on the existence and uniqueness of fixed points in partially ordered metric spaces for monotone mappings by removing the assumption of continuity. We adapt the notion of tripled fixed points so that the solution (X, Y, Z), $X = Y = Z$ of the system represents a solution of

$$X = Q + p_1 A^* X^{-k_1} A + p_2 B^* Y^{-k_2} B + p_3 C^* Z^{-k_3} C.$$

In contrast to the previously known results for tripled fixed points of maps with or without the mixed monotone property in partially ordered complete metric spaces, we demonstrate that it is possible to obtain results for the existence and uniqueness of such points for arbitrary maps with a type of monotonicity, with the partial ordering in the Cartesian product arising from the maps itself. We prove theorems that ensure the existence and uniqueness for tripled fixed points for maps with different types of monotone properties. We obtain sufficient conditions for the existence and uniqueness of systems of three nonlinear matrix equations.

3. Application of the N-tupled fixed point theorem in a partially ordered metric space to find a positive definite solution to a system of matrix equations.

Finding a connection between matrix equations or systems and the partial ordering they give rise to in the considered Cartesian products, which allows to expand the class of problems solvable with the theory of N-orc fixed points.

Using deep results of M. Berzig, X. F. Duan, B. Samet, R.M. Bianchini, M. Grandolfi, T. G. Bhaskar, V. Lakshmikantham, A. C. M. Ran, M. C. B. Reurings and B. Samet, C. Vetro we tried to unify the techniques for fixed points, fixed point pairs, fixed point triples, and N-tupled fixed points for weakly monotonic maps, i.e. maps that have some kind of monotonicity for each of their variables.

A-5 ***A. Ali, M. Hristov, H. Kulina, A. Ilchev, B. Zlatanov, Applications of N-Tupled Fixed Points in Partially Ordered Metric Spaces for Solving Systems of Nonlinear Matrix Equations Mathematics 2025, 13(13), 2125; https://doi.org/10.3390/math13132125, ISSN:2227-7390,*** **[Г-7.3.]**.

In this paper, we weaken the classical requirement for contraction in partially ordered metric spaces by requiring it to be satisfied only for the iteration sequence generated by the image, and not for every pair of comparable elements. We replace the continuity requirement with a weaker condition that is sufficient for the existence and uniqueness of fixed points. In this way, we generalize the main results of A. C. M. Ran, M. C. B. Reurings. Following the observations from the previous three articles [A-1], [A-2] and [A-4] in [A-5] we show that the type of monotonicity in the different variables of the considered N-tupled maps $(F_1, F_2, \dots F_N)$ gives rise to the definition of the partial order in the Cartesian work under consideration. Using the idea of B. Samet, C. Vetro to define the N-tupled fixed points for map, but we generalize it to N-tupled maps $(F_1, F_2, \dots F_N)$, and with a special choice of the maps F_i, $i = 2, 3, \dots, N$ we obtain the results of B. Samet, C. Vetro. This approach allows the elements in the ordered N-tupled fixed points not to be necessarily equal to each other.

The results mentioned as contributions in the previous points are a fair case of the results we obtained here. Also, other well-known results are consequences of our results for N-tupled fixed points and our results on the existence and uniqueness of solutions of systems of matrix equations.

We have chosen to prove the main result for N-tupled fixed points for an ordered N-tupled of maps in the classical way, instead of applying our theorem, which generalizes A. C. M. Ran, M. C. B. Reurings and identify N-tupled fixed points for an ordered N-tupled of maps $(F_1, F_2, \dots F_N)$ with the proposed B. Samet, C. Vetro fixed point for $T(\overline{x}) = (F_1\overline{x}, F_2\overline{x}, \dots F_N\overline{x})$, where $\overline{x} = (x_1, x_2, \dots, X_N)$. We obtain sufficient results for the existence and uniqueness of solutions to various matrix equations.

Shumen
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