On Some Families of Multi-point Iterative methods for Solving Nonlinear Equations

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Abstract

Some semi-discrete analogous of well known one-point family of iterative methods for solving nonlinear scalar equations dependent on an arbitrary constant are proposed. The new families give multi-point iterative processes with the same or higher order of convergence. The convergence analysis and numerical examples are presented.

Keywords: Iterative function, Order of convergence, One-point iterative process, Multi-point iterative process.
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1 Introduction

We consider the nonlinear equation

$$f(x) = 0\,, \tag{1.1}$$

and let $\alpha \in (a, b)$ be a simple root of f and $f \in C^2[a, b]$ for a sufficiently small interval $[a, b]$. There are many known iterative methods and families of iterative methods for solving equation (1.1). Let us consider one of the best-known families- the Chebyshev-Halley family

$$\varphi(x) = x - \frac{f(x)}{f'(x)}\left(1 + \frac{1}{2}\,\frac{T_f(x)}{1 - \lambda T_f(x)}\right), \qquad T_f(x) = \frac{f(x)f''(x)}{f'(x)^2}\,. \tag{1.2}$$

Here λ is an arbitrary real parameter. This family has third order of convergence. Particular cases are: Chebyshev's method (CM) for $\lambda = 0$; Halley's method (HM) for $\lambda = \frac{1}{2}$ and Super-Halley method (SHM) for $\lambda = 1$. When $\lambda \to \pm\infty$, we get Newton's method. The family (1.2) was studied by J. M. Gutierrez and M. A. Hernandez in [1] and J. L. Varona in [2]. These methods also can be seen in [3, 4, 8].

Our purpose here is to construct some corresponding to (1.2) families which do not require computation of the second derivative of the function f. Thus, the new families are intended to unify multi-point iterative formulae. Similar approach and iterative formulae have been considered in [5, 7].

In *Section 2*, we construct two new families. Particularly from these two families we obtain known and new iterative formulae. Moreover a fourth order iterative method is obtained. *Section 3* includes the analysis of convergence for the methods presented. Some numerical examples are shown in *Section 4* and the conclusion in *Section 5*.

2 Families of Multi-point iterative methods

In this section we derive two new families.

2.1 First family

Let us use the following Taylor expansion for the function $f(x - u)$ around the point x, where $f(x) \neq 0$

$$f(x - u) = f(x) - uf'(x) + \frac{u^2}{2}f''(x) + O(u^3) = \frac{u^2}{2}f''(x) + O(u^3)\,.$$

Here we denote $u = u(x) = \frac{f(x)}{f'(x)}$. Thus we get

$$\begin{aligned} T_f(x) &= \frac{f(x)f''(x)}{(f'(x))^2} = \frac{2}{f(x)} \left(\frac{f(x)}{f'(x)}\right)^2 \frac{f''(x)}{2} \\ &= \frac{2}{f(x)} \frac{u^2 f''(x)}{2} \approx \frac{2\, f(x-u)}{f(x)} . \end{aligned}$$

Substituting this expression in (1.2), we get the corresponding multi-point family

$$\varphi(x) = x - \frac{f(x)}{f'(x)} \left(1 + \frac{1}{2} \frac{\frac{2f(x-u)}{f(x)}}{1 - 2\lambda \frac{f(x-u)}{f(x)}} \right), \quad \text{where} \quad u = \frac{f(x)}{f'(x)} .$$

It is equivalent to

$$\varphi(x) = x - \frac{f(x)}{f'(x)} \left(1 + \frac{f(x-u)}{f(x) - 2\lambda f(x-u)} \right), \quad \text{where} \;\; u = \frac{f(x)}{f'(x)} . \tag{2.1}$$

This family also depends on real arbitrary parameter λ, it will be studied in Theorem 3.1. The corresponding particular cases are:

1) For $\lambda = 0$, we get the formula

$$\varphi(x) = x - \frac{f(x)}{f'(x)} - \frac{f(x-u)}{f'(x)} . \tag{2.2}$$

This iterative formula is explored by J. F. Traub (see [3], pp. 142), and it is of third order of convergence.

2) For $\lambda = \frac{1}{2}$ we get

$$\varphi(x) = x - \frac{f^2(x)}{f'(x)(f(x) - f(x-u))} . \tag{2.3}$$

This is the Newton-Secant formula (see [3], pp. 146, [5, 7]), also of third order of convergence.

3) For $\lambda = 1$ we get

$$\varphi(x) = x - \frac{f(x)}{f'(x)} \left(\frac{f(x) - f(x-u)}{f(x) - 2f(x-u)} \right) . \tag{2.4}$$

This is the Traub-Ostrowski formula, which is an order four formula (see [3], pp. 150).

4) When $\lambda \to \pm\infty$, we get Newton's method with second order of convergence.

A modification of the presented family (2.1) for simultaneous extraction of all roots of algebraic polynomial is proposed in [6].

2.2 Second family

Using the approximation

$$f''(x) \approx \frac{f'(x) - f'(x - \beta u)}{\beta u},$$

where $\beta \in \mathbf{R}$, and substitution in formula (1.2), we get the new family

$$\varphi(x) = x - u(x)\left(1 + \frac{1}{2}\,\frac{f'(x) - f'(x - \beta u(x))}{(\beta - \lambda)f'(x) + \lambda f'(x - \beta u(x))}\right). \tag{2.5}$$

This method will be studied in Theorem 3.2. Some interesting particular cases are:

1) For $\beta = 1$, then for $\lambda = \frac{1}{2}$, we get

$$\varphi(x) = x - \frac{2f(x)}{f'(x) + f'(x - u(x))}. \tag{2.6}$$

This formula is obtained by J. Traub (see [3], pp. 134) and by S. Weerakoon and T. Fernando (see [9]). It has third order of convergence.

2) For arbitrary $\lambda = \beta \in \mathbf{R}$, we obtain the formula

$$\varphi(x) = x - \frac{2\beta - 1}{2\beta}\,u(x) - \frac{1}{2\beta}\,\frac{f(x)}{f'(x - \beta u(x))}. \tag{2.7}$$

For $\beta = \frac{1}{2}$, then for $\lambda = \frac{1}{2}$, we get

$$\varphi(x) = x - \frac{f(x)}{f'\left(x - \frac{u(x)}{2}\right)}. \tag{2.8}$$

This process has third order of convergence and it is one of the best known multi-point methods (see [3], pp. 134).

For $\beta = 1$ then for $\lambda = 1$, we get

$$\varphi(x) = x - \frac{f(x)}{2}\left(\frac{1}{f'(x)} + \frac{1}{f'(x - u(x))}\right). \tag{2.9}$$

That is also third order iterative process, explored by J. Traub (see [3], pp. 135).

3) For $\lambda = 0$ and $0 \neq \beta \in \mathbf{R}$, we get

$$\varphi(x) = x - \frac{2\beta + 1}{2\beta}\,u(x) + \frac{1}{2\beta}\,\frac{f'(x - \beta u(x))}{f'(x)}\,u(x). \tag{2.10}$$

For $\beta = -\frac{1}{2}$ ($\lambda = 0$), we obtain

$$\varphi(x) = x - \frac{f'\left(x + \frac{u(x)}{2}\right)}{f'(x)} \, u(x) \,. \tag{2.11}$$

4) For $\beta = \frac{2}{3}$, then for $\lambda = 1$, we obtain

$$\varphi(x) = x - \frac{u(x)}{2} \left(\frac{3f'(y) + f'(x)}{3f'(y) - f'(x)} \right) , \tag{2.12}$$

where $y = x - \frac{2}{3} \, u(x)$. This is an order four method (see Theorem 3.2). The formulae (2.7), (2.10), (2.11) and (2.12) generate new iterative methods for solving nonlinear equations. We believe that such cases are not considered by other authors.

2.3 Some other formulae

We can use some other approximations of the second derivative of the function f in the family (1.2) to get some new formulae of iterative processes.

By means of an analogous procedure we can substitute the second derivative f'' of the function f with different approximations, such as the following formulae

$$f''(x) \approx \frac{f'(x+u) - f'(x)}{u} \,; \; f''(x) \approx \frac{f'(x+u) - f'(x-u)}{2u} \,;$$

$$f''(x) \approx \frac{2(f'(x) - f'(x - \frac{u}{2}))}{u} \,.$$

Thus, we obtain different multi-point families. For example, if we use the approximation

$$f''(x) \approx \frac{5f'(x) - 4f'(x - \frac{u}{2}) - f'(x-u)}{3u}$$

then for $\lambda = \frac{1}{2}$ from family (1.2) we get

$$\varphi(x) = x - \frac{6f(x)}{f'(x) + 4f'\left(x - \frac{u}{2}\right) + f'(x-u)}.$$

This formula is studied by V. Hasanov, I. Ivanov and G. Nedzhibov (see [10]) and it has third order of convergence.

2.4 Notes and comparison of the families

In order to compare the families observed, we will use the following criteria: *informational usage* (d) and *efficiency index* (*EFF*). The *informational usage* is equal to the quantity of computations of the values of the function f and its derivatives in each iteration. The *efficiency index* is the ratio:

$$EFF = \frac{\rho}{d},$$

where ρ is the order of convergence of the corresponding method, see [3], pp. 17.

- Compared with the family (1.2) where on each one step of iteration it does requires the computation of f, f' and f'', the formulae and families presented in *Section 2* do require the computation only of the values f and f'. For the one-point family the *informational usage* is $d = 3$ (see [3], pp. 17–18) and for the multi-point formulas it is $d = 2$ (see [3], pp. 17–18).

- The number of arithmetic operations on family (1.2) are equal to ten (seven operations by priority of multiplication or division and three operations by priority of addition or subtraction). As far as the operations on family (2.1) are concerned, they are equal to eight (five by priority of multiplication and three by priority of addition) and on family (2.5) are equal to eight (four by priority of multiplication and four by priority of addition).

- Since the order of convergence of family (1.2) is three for all values of the parameter $\lambda\,(\lambda \neq \infty)$, then the *efficiency index* is $EEF = 1$. Family (2.1) has an order three for $\lambda \neq 1$, and order four for $\lambda = 1$, hence the *efficiency index* is $EEF = \frac{3}{2}$ and $EEF = 2$ respectively. For the family (2.5) the *efficiency index* is $EEF = \frac{3}{2}$ for $(\lambda, \beta) \neq (1, 2/3)$, and $EEF = 2$ for $(\lambda, \beta) = (1, 2/3)$.

- Consequently, for $\lambda = 1$ and for sufficiently small neighbourhood of $\lambda = 1$, and for $(\lambda, \beta) = (1, 2/3)$ and sufficiently small neighbourhood of $(\lambda, \beta) = (1, 2/3)$ the methods obtained by the families (2.1) and (2.5), respectively have higher efficiency and have higher order of convergence than the corresponding methods obtained by the family (1.2).

- The two families discussed (2.1) and (2.5) unify some of the most known multi-point iterative methods for solving nonlinear equation. Meanwhile they provide some more unknown processes.

3 Analysis of convergence

The following two theorems concern the order of convergence of the families (2.1) and (2.5):

Theorem 3.1 *Let $f : A \to R$ is defined, continuous and enough time differentiable in the open interval A. If $f(x)$ has a simple root $\alpha \in A$, then for sufficiently close initial approximation to α the family (2.1) has an order of convergence*

(i) *3, for $\lambda \neq 1$;*

(ii) *4, for $\lambda = 1$.*

Proof: We denote $\varepsilon = x - \alpha,\ \delta = \varphi(x) - \alpha,\ u = u(x) = \frac{f(x)}{f'(x)}\,,\ C_2 = \frac{f''(\alpha)}{2f'(\alpha)}\,.$ Let us use the Taylor expansions

$$\begin{aligned} f(x) &= f(\alpha) + \varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha) + O(\varepsilon^3) \\ &= \varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha) + O(\varepsilon^3)\,, \qquad \text{and} \\ f'(x) &= f'(\alpha) + \varepsilon f''(\alpha) + O(\varepsilon^2). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \frac{f(x)}{f'(x)} &= \frac{\varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha) + O(\varepsilon^3)}{f'(\alpha) + \varepsilon f''(\alpha) + O(\varepsilon^2)} \\ &= \left[\varepsilon + \varepsilon^2 C_2 + O(\varepsilon^3)\right]\left[1 - 2\varepsilon C_2 + O(\varepsilon^2)\right] \\ &= \varepsilon - \varepsilon^2 C_2 + O(\varepsilon^3)\,. \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon - u &= \varepsilon^2 C_2 + O(\varepsilon^3)\,, \\ f(x-u) &= (\varepsilon - u) f'(\alpha) + \frac{(\varepsilon - u)^2}{2} f''(\alpha) + O\left((\varepsilon - u)^3\right)\,. \end{aligned}$$

Then

$$\begin{aligned} \frac{f(x-u)}{f(x)} &= \frac{(\varepsilon - u) f'(\alpha) + \frac{(\varepsilon-u)^2}{2} f''(\alpha) + O\left((\varepsilon - u)^3\right)}{\varepsilon f'(\alpha) + \frac{\varepsilon^2}{2} f''(\alpha) + O(\varepsilon^3)} \\ &= \left[\frac{(\varepsilon - u)}{\varepsilon} + \frac{(\varepsilon - u)^2}{\varepsilon} C_2 + O\left(\frac{(\varepsilon - u)^3}{\varepsilon}\right)\right]\left[1 - \varepsilon C_2 + O\left(\varepsilon^2\right)\right] \\ &= \frac{(\varepsilon - u)}{\varepsilon} - (\varepsilon - u) C_2 + O(\varepsilon^3) \\ &= \frac{(\varepsilon - u)}{\varepsilon} + O(\varepsilon^2) \qquad (3.1) \end{aligned}$$

and

$$\begin{aligned}\frac{f(x-u)}{f'(x)} &= \frac{(\varepsilon-u)f'(\alpha)+\frac{(\varepsilon-u)^2}{2}f''(\alpha)+O\left((\varepsilon-u)^3\right)}{f'(\alpha)+\varepsilon f''(\alpha)+O(\varepsilon^2)}\\ &= \left[(\varepsilon-u)+(\varepsilon-u)^2C_2+O\left((\varepsilon-u)^3\right)\right]\left[1-2\varepsilon C_2+O(\varepsilon^2)\right]\\ &= (\varepsilon-u)-2\varepsilon(\varepsilon-u)C_2+O(\varepsilon^4)\,. \qquad (3.2)\end{aligned}$$

Finally, from the equation (2.1) we have

$$\delta=\varepsilon-u-u\frac{f(x-u)}{f(x)-2\lambda f(x-u)}=(\varepsilon-u)-\frac{\frac{f(x-u)}{f'(x)}}{1-2\lambda\frac{f(x-u)}{f(x)}}\,.$$

Substituting (3.1) and (3.2), we get

$$\delta=(\varepsilon-u)-\frac{(\varepsilon-u)-2\varepsilon(\varepsilon-u)C_2+O(\varepsilon^4)}{1-2\lambda\left(\frac{(\varepsilon-u)}{\varepsilon}+O(\varepsilon^2)\right)}.$$

Since λ is a constant, then for a sufficiently small ε we obtain

$$\begin{aligned}\delta &= (\varepsilon-u)-\left[(\varepsilon-u)-2\varepsilon(\varepsilon-u)C_2+O(\varepsilon^4)\right]\left[1+2\lambda\frac{(\varepsilon-u)}{\varepsilon}+O(\varepsilon^2)\right]\\ &= 2C_2(\varepsilon-u)\varepsilon-2\lambda\frac{(\varepsilon-u)^2}{\varepsilon}+O(\varepsilon^4)\\ &= 2C_2^2\varepsilon^3-2\lambda C_2^2\varepsilon^3+O(\varepsilon^4)\\ &= 2(1-\lambda)C_2^2\varepsilon^3+O(\varepsilon^4)\,.\end{aligned}$$

We have shown that for $\lambda\neq 1$ the estimate $\delta=O(\varepsilon^3)$ holds, that proves assertion (i), and for $\lambda=1$, $\delta=O(\varepsilon^4)$, which proves assertion (ii). □

Theorem 3.2 *Let $f:A\to R$ be continuous and higher enough times differentiable in the open interval A. If $f(x)$ has a simple root $\alpha\in A$, then for sufficiently close initial approximation to α the family (2.5) has an order of convergence*

(i) 3, for $(\lambda,\beta)\neq\left(1,\frac{2}{3}\right)$;

(ii) 4, for $\lambda=1$ and $\beta=\frac{2}{3}$.

Proof: We denote

$$u=u(x)=\frac{f(x)}{f'(x)},\ \varepsilon=x-\alpha,\ \delta=\varphi(x)-\alpha,\ C_2=\frac{f''(\alpha)}{2f'(\alpha)},\ C_3=\frac{f'''(\alpha)}{6f'(\alpha)}\,.$$

From expression (2.5), we obtain

$$\varphi(x) \quad = \quad x - u(x)\left(1 + \frac{\frac{1}{2\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}{1 - \frac{\lambda}{\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}\right)$$

or the equivalent equation

$$\begin{aligned}\delta = \varphi(x) - \alpha \quad &= \quad \varepsilon - u(x)\left(1 + \frac{\frac{1}{2\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}{1 - \frac{\lambda}{\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}\right) \\ &= \quad \varepsilon - u(x)\left(\frac{1 - \frac{2\lambda-1}{2\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}{1 - \frac{\lambda}{\beta}\left(1 - \frac{f'(x-\beta u(x))}{f'(x)}\right)}\right) .\end{aligned}$$

From the expressions

$$f'(x) = f'(\alpha)(1 + 2C_2\varepsilon + 3C_3\varepsilon^2 + O(\varepsilon^3))$$

and

$$f'(x - \beta u) = f'(\alpha)(1 + 2(1-\beta)C_2\varepsilon + (2\beta C_2^2 + 3(1-\beta)^2 C_3)\varepsilon^2 + O(\varepsilon^3))\,,$$

we obtain

$$\frac{f'(x - \beta u)}{f'(x)} = 1 - 2\beta C_2\varepsilon + (6\beta c_2^2 - 3(2\beta - \beta^2)C_3)\varepsilon^2 + O(\varepsilon^3).$$

Using

$$u = \varepsilon - C_2\varepsilon^2 + 2(C_2^2 - C_3)\varepsilon^3 + O(\varepsilon^4)\,,$$

and substituting in the last expression of δ, finally we get

$$\delta \quad = \quad \left(2(1-\lambda)C_2^2 + \frac{3\beta - 2}{2}\,C_3\right)\varepsilon^3 + O(\varepsilon^4).$$

We have shown that for arbitrary values of the parameters λ and β, such that

$$(\lambda, \beta) \neq \left(1, \frac{2}{3}\right)\,,$$

the estimate

$$\delta = O(\varepsilon^3)$$

is satisfied, which proves assertion (i), and for $(\lambda, \beta) = \left(1, \frac{2}{3}\right)$ the estimate $\delta = O(\varepsilon^4)$ holds, which proves assertion (ii). □

4 Numerical Experiments

We have done numerical experiments with different functions and initial approximations. All programs were realized in MATLAB. We compare the iterative methods observed on the base of following criteria: *number of iterations*, *order of convergence* and *absolute error*. The initial points are chosen in an open interval $U(\alpha)$, where the first and second derivatives of the function retain their signs.

We use the following stoping tests for computer programs:

$$\begin{array}{ll} 1) & \frac{|x_n-\alpha|}{|x_n|} < tol; \\ 2) & |f(x_n)| < tol\,, \end{array}$$

where the both of them should be satisfied.

The computational order of convergence ρ is approximated using the formula:

$$\rho = \ln\left|\frac{x_{n+1}-\alpha}{x_n-\alpha}\right| \Big/ \ln\left|\frac{x_n-\alpha}{x_{n-1}-\alpha}\right| \,,$$

where x_i is the approximation number i of the root α (for $n+1 = iter$) and tol = `1e-15`. The expression of ρ is obtained using the definition of order of convergence, see [10] and [3], pp. 15.

We introduce the notations: an initial point x_0; number of iterations for which 1) and 2) are satisfied $iter$; computational order of convergence ρ; the absolute error err, computed by the formula $|x_n - \alpha|$ (for $n = iter$); Newton's method NM; Newton-Secant method NSM, see (2.3); Super-Halley method SHM; Halley's method HM; Chebyshev's method CM.

In the following examples when the number of iterations $iter \le 3$ or when the exactly root α is reached for $iter \le 4$, the computational order of convergence ρ is not computed.

First example. Let us consider the function $f(x) = (x-2.01)(x-2.02)(x-2.03)(x-2.04)(x-2.05)(x-2.06)(x-3.01)(x-3.02)(x-3.03)\times(x-3.04)$. For initial approximation of the roots $\alpha_1 = 2.01; \alpha_2 = 2.02$ and $\alpha_3 = 3.01$ we use the points $x_0 = 2.013; x_0 = 2.022$ and $x_0 = 2.8$, respectively . The results are represented in Table 1.

Table 1. Experimental results of Example first.

	$x_0 = 2.013$		$x_0 = 2.022$		$x_0 = 2.8$	
$\varphi(x)$	iter	ρ	iter	ρ	iter	ρ
NM	9	1.98	4	1.99	12	1.98
NSM	8	3.02	3	-	8	2.98
(2.11)	5	2.99	3	-	9	2.99
(2.12)	4	3.73	2	-	6	3.81

Second example. Let $f(x) = (x-2)^6(x-3)^4$ with multiple roots $\alpha_1 = 2$ and $\alpha_2 = 3$. The initial approximations are $x_0 = 2.1; x_0 = 1.7$ and $x_0 = 3.5$, see the results in Table 2.

Table 2. Experimental results of Example second.

	$x_0 = 2.1$		$x_0 = 1.7$		$x_0 = 3.5$	
$\varphi(x)$	iter	ρ	iter	ρ	iter	ρ
NM	19	0.99	27	1.00	31	0.99
NSM	12	0.99	17	1.00	20	0.99
(2.11)	12	0.99	17	1.00	20	0.99
(2.12)	9	0.99	12	1.00	14	0.99

Third example. For $f(x) = x^3+4x^2-6+\cos(x-1)$ with root $\alpha = 1$ and the interval $(\frac{1}{2}, 3)$, where the function is convex and increasing. The initial approximations are $x_0 = 1.8$ and $x_0 = 3.2$ (see Table 3). For this example we have used both stoping tests: 1) $|x_{n+1} - x_n| < tol$ and 2) $|f(x_{n+1})| < tol$, with $tol = \sqrt{\varepsilon}$ (where $\varepsilon = 2.22$ `e-16` is a MATLAB constant).

Table 3. Experimental results of Example third.

		$x_0 = 1.8$			$x_0 = 3.2$		
λ		iter	ρ	err	iter	ρ	err
$\lambda = 1$	(2.4)	3	-	0	3	3.93	1.02e-14
	(SHM)	3	-	0	3	2.98	6.66e-16
$\lambda = \frac{1}{2}$	(2.3)	3	2.88	6.50e-13	4	2.96	2.22e-16
	(HM)	3	2.99	2.79e-14	4	-	0
$\lambda = 0$	(2.2)	3	2.95	1.82e-10	4	2.97	5.83e-12
	(CM)	3	2.96	5.01e-11	4	2.98	4.98e-12

5 Conclusion

This study represents several formulas and families of multi-point iterative methods for solving nonlinear scalar equations which are analogous to a known family of one-point methods. In particular cases from the families obtained, we get some well known methods. Unlike the one-point family, the new families do not require computation of the second derivative of the f function, but the number of computations of the value of function f or first derivative of the function f increases with one more, on each step of iteration. The results from numerical experiments confirm the assertions.

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