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NOETHER'S PROBLEM FOR PERMUTATIONAL WREATH PRODUCTS

Ivo M. Michailov

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Abstract

Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. In Noether's problem then it is asked whether $K(G)$ is rational over K. In this paper we prove two reduction theorems concerning Noether's problem for permutational wreath products.

Key words: Noether's problem, the rationality problem, hyperoctahedral group, wreath product

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1. Introduction. Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G$. *Noether's problem* then asks whether $K(G)$ is rational (= purely transcendental) over K. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions (see [1,2] and [3], 33.1, p. 86).

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The following well-known theorem gives a positive answer to the Noether's problem for abelian groups.

Theorem 1.1 (Fischer ([1], Theorem 6.1)). *Let G be a finite abelian group of exponent e. Assume that* (i) *either char $K = 0$ or char $K > 0$ with char $K \nmid e$, and* (ii) *K contains a primitive e-th root of unity. Then $K(G)$ is rational over K.*

SWAN's paper [1] also gives a survey of many results related to the Noether's problem for abelian groups. In the same time, just a handful of results about Noether's problem are obtained when the groups are non abelian. Kang and Plans prove the following reduction theorem concerning wreath products.

Theorem 1.2 ([4], Theorem 1.10). *Let K be any field, $H \wr G$ be the wreath product of finite groups H and G. If $K(H)$ is rational (resp. stably rational) over K, so is $K(H \wr G)$ over $K(G)$.*

The purpose of this paper is to extend Theorem 1.2 for the permutational wreath products. First, recall the definitions of the regular and the permutational wreath products.

Definition 1. Let H and G be finite groups. Define $N = \bigoplus_{g\in G} H_g$, where each H_g is a copy of H. If $\sigma \in G$ and $x = (\cdots, x_g, \cdots) \in N$, define ${}^{\sigma}x = (\cdots, x_{\sigma^{-1}g}, \cdots)$. The (regular) wreath product $H \wr G$ is a semi-direct product $N \rtimes G$.

Definition 2. Let H and G be finite groups with G a subgroup of the symmetric group S_n. Define $N = \bigoplus_{i=1}^{n} H_i$, where each H_i is a copy of H. If $\sigma \in G$ and $x = (x_1, \cdots, x_n) \in N$, define ${}^{\sigma}x = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. The permutational wreath product $H\mathrm{pwr}G$ is the semi-direct product $N \rtimes G$.

More precisely, for both products we have that if $x, y \in N$ and $\sigma, \tau \in G$, then $(x, \sigma) \cdot (y, \tau) = (x \cdot ({}^{\sigma}y), \sigma\tau)$. Thus we have $(\sigma x)(\tau y) = (\sigma\tau)({}^{\tau^{-1}}x \cdot y)$.

Note that the order of the (regular) wreath product is $|H|^{|G|}|G|$, whereas the order of the permutational wreath product is $|H|^n|G|$. Our first main result is the following

Theorem 1.3. *Let H and G be finite groups with G a transitive subgroup of the symmetric group S_n; let K be any field, and let $H\mathrm{pwr}G$ be the permutational wreath product. If $K(H)$ is rational over K, so is $K(H\mathrm{pwr}G)$ over $K(G)$.*

It is well known that $K(S_n)$ is rational over any K for any $n > 1$. Therefore, $K(C_2\mathrm{pwr}S_n)$ is rational over K, where $C_2\mathrm{pwr}S_n$ is known as the *hyperoctahedral group.*

Theorem 1.3 can be generalized for non-transitive groups as well. We will prove a somewhat simplified version of this result in the following

Corollary 1.4. *Let H and G be finite groups with G a subgroup of the symmetric group S_n; let K be any field, and let $H\mathrm{pwr}G$ be the permutational wreath product. If $K(H)$ and $K(G)$ are rational over K, so is $K(H\mathrm{pwr}G)$ over K.*

We will need the following well known result

Theorem 1.5 ([5], Theorem 1)**.** *Let G be a finite group acting on $L(x_1,\dots,x_m)$, the rational function field of m variables over a field L such that*

(i) *for any $\sigma\in G, \sigma(L)\subset L$;*

(ii) *the restriction of the action of G to L is faithful;*

(iii) *for any $\sigma\in G$,*

$$\begin{pmatrix}\sigma(x_1)\\ \vdots\\ \sigma(x_m)\end{pmatrix}=A(\sigma)\begin{pmatrix}x_1\\ \vdots\\ x_m\end{pmatrix}+B(\sigma),$$

where $A(\sigma)\in \mathrm{GL}_m(L)$ and $B(\sigma)$ is $m\times 1$ matrix over L. Then there exist $z_1,\dots,z_m\in L(x_1,\dots,x_m)$ so that $L(x_1,\dots,x_m)^G=L^G(z_1,\dots,z_m)$ and $\sigma(z_i)=z_i$ for any $\sigma\in G$, any $1\le i\le m$.

2. Proof of Theorem 1.3. Write $\widetilde{G}=H\mathrm{pwr}G$. Define a group isomorphism $\phi_i: H\to H_i$ such that $\phi_i(h)=(1,\dots,\underset{i}{h},\dots,1)$ where $H_i\cong\{(1,\dots,\underset{i}{h},\dots,1)|h \in H\}$.

Define a subgroup $M=\bigoplus_{i=2}^n H_i$ of N. Note that the coset decomposition of $\widetilde{G}$ with respect to M is given as $\widetilde{G}=\cup_{\sigma\in G,h\in H}(\sigma\cdot\phi_1(h))M$.

Let $V=\bigoplus_{g\in G}K\cdot u(g)$ and $W=\bigoplus_{x\in N}K\cdot u(x)$ be the regular representation spaces of G and N respectively. Define an action of $\widetilde{G}$ on $V\bigotimes_K W$ by $(gx)\cdot(u(g')\otimes u(y))=u(gg')\otimes u({}^{g'^{-1}}x\cdot y)$, where $g,g'\in G$ and $x,y\in N$. It follows that $V\bigotimes_K W$ is isomorphic to the regular representation space of $\widetilde{G}$.

Now, let $W_i=\bigoplus_{h\in H}K\cdot u(\phi_i(h))$ be the regular representation space for H_i, $1\le i\le n$. Then we can regard $\bigotimes_{i=1}^n W_i$ as the regular representation space of N, i.e., W. Similarly, we can regard $\bigotimes_{i=2}^n W_i$ as the regular representation space of M. Define

$$w_i=\sum_{h\in H}u(\phi_i(h))\in W_i, w'=\otimes_{i=2}^n w_i\in\bigotimes_{i=2}^n W_i,$$

$$w_0=u(1)\otimes w'\in W, u_0=u(1)\otimes w_0\in V\bigotimes_K W.$$

Note that $x\cdot u_0=u_0$ for any $x\in M$.

For any $g\in G, h\in H$, define

$$u(g;h)=(g\cdot\phi_1(h))\cdot u_0=u(g)\otimes(u(\phi_1(h))\otimes w')\in V\bigotimes_K W.$$

Note that for any $g,g'\in G$ and $h\in H$, we have $g\cdot u(g';h)=u(gg';h)$.

Now choose and fix arbitrary $i : 1 \leq i \leq n$. Since G is transitive, there exists $g \in G$ such that $g(1) = i$. Denote by $\mathrm{St}(s)$ the stabilizer of any $s:\ 1 \leq s \leq n$, i.e. the subgroup $\{\sigma \in G | \sigma(s) = s\}$. Then for any $h, h' \in H$ and $g' \in g\mathrm{St}(1)$, we have

$$\phi_i(h) \cdot u(g'; h') = g' \cdot ({}^{g'^{-1}}\phi_i(h)\phi_1(h')) \cdot u_0 = u(g'; hh').$$

If $g' \notin g\mathrm{St}(1)$, we have $\phi_i(h) \cdot u(g'; h') = u(g'; h')$.

For each $g \in G$, define

$$U_g = \bigoplus_{h \in H} K \cdot u(g; h) \subset V \bigotimes_K W,\ \widetilde{U} = \bigoplus_{g \in G} U_g \subset V \bigotimes_K W.$$

Note that G permutes the spaces U_g regularly, H_i acts regularly on U_g if $g(1) = i$, and H_i acts trivially on U_g if $g(1) \neq i$. Taking into account that G is transitive, we obtain that $\widetilde{U}$ is a faithful $\widetilde{G}$ subspace of $V \bigotimes_K W$.

Next, apply Theorem 1.5. We find that $K(\widetilde{G})$ is rational over $K(\widetilde{U})^{\widetilde{G}}$. It remains to show that $K(\widetilde{U})^{\widetilde{G}}$ is rational over $K(G)$, provided that $K(H)$ is rational over K.

Let $K(H)$ be rational over K. Then for any $g \in G$ such that $g(1) = i$ we have that $K(U_g)^{H_i}$ is rational over K. Choose a transcendence basis $\{v(g; j) \mid 1 \leq j \leq |H|\}$ for $K(U_g)^{H_i}$, i.e., we may write $K(U_g)^{H_i} = K(v(g; j) : 1 \leq j \leq |H|)$. Since G permutes regularly U_g, we can write $\sigma \cdot v(g; j) = v(\sigma g; j)$.

Now write the coset decomposition of G with respect to $\mathrm{St}(1) : G = \cup_{i=1}^{k} g_i \mathrm{St}(1)$, where $(G : \mathrm{St}(1)) = k$. Thus

$$K(\widetilde{U})^{\widetilde{G}} = (K(\widetilde{U})^N)^G = \left(\cup_{i=1}^{k} K(v(g; j) : 1 \leq j \leq |H|, g \in g_i\mathrm{St}(1))\right)^G.$$

Then from Theorem 1.5 follows that $K(\widetilde{U})^{\widetilde{G}}$ is rational over

$$\left(\cup_{i=1}^{k} K(v(g; 1) : g \in g_i\mathrm{St}(1))\right)^G = K(v(\sigma; 1) : \sigma \in G)^G = K(G).$$

3. Proof of Theorem 1.4. Write again $\widetilde{G} = H\mathrm{pwr}G$. Let $O(i_1), \ldots, O(i_m)$ be the orbits of G, i.e., $O(i_j) = \{\sigma(i_j) \mid \sigma \in G\}$ and $\{1, \ldots, n\} = O(i_1) \cup \cdots \cup O(i_m)$. We can assume that $|O(i_j)| > 1$ for any $j = 1, \ldots, m$.

Define a subgroup $M_j = \bigoplus_{i \neq i_j} H_i$ of N for any $j = 1, \ldots, m$. The coset decomposition of $\widetilde{G}$ with respect to M_j is given as $\widetilde{G} = \cup_{\sigma \in G, h \in H}(\sigma \cdot \phi_{i_j}(h))M_j$.

Define the spaces V, W and W_i in the same way as in the proof of Theorem 1.3. We can regard $\bigotimes_{i\neq i_j} W_i$ as the regular representation space of M_j. Define

$$w_i = \sum_{h\in H} u(\phi_i(h)) \in W_i,$$

$$w_{j0} = w_1 \otimes \cdots \otimes \underset{i_j}{u(1)} \otimes \cdots \otimes w_n \in W, u_{j0} = u(1) \otimes w_{j0} \in V \bigotimes_K W.$$

Note that $x \cdot u_{j0} = u_{j0}$ for any $x \in M_j$.

For any $g \in G, h \in H$, define

$$u_j(g;h) = (g \cdot \phi_{i_j}(h)) \cdot u_{j0} = u(g) \otimes (w_1 \otimes \cdots \otimes \underset{i_j}{u(\phi_{i_j}(h))} \otimes \cdots \otimes w_n) \in V \bigotimes_K W.$$

Note that for any $g, g' \in G$ and $h \in H$, we have $g \cdot u_j(g';h) = u_j(gg';h)$.

Now choose and fix arbitrary $s : 1 \leq s \leq n$. Then there exists $g \in G$ and $j : 1 \leq j \leq m$, such that $g(i_j) = s$. Then for any $h, h' \in H$ and $g' \in g\mathrm{St}(i_j)$, we have

$$\phi_s(h) \cdot u_j(g';h') = g' \cdot ({}^{g'^{-1}}\phi_s(h)\phi_{i_j}(h')) \cdot u_0 = u_j(g';hh').$$

If $g' \notin g\mathrm{St}(i_j)$, we have $\phi_s(h) \cdot u_j(g';h') = u_j(g';h')$.

For each $j : 1 \leq j \leq m$ and each $g \in G$, define the subspaces of $V \bigotimes_K W$

$$U_g^{(j)} = \bigoplus_{h\in H} K \cdot u_j(g;h),\ \widetilde{U}^{(j)} = \bigoplus_{g\in G} U_g^{(j)},\ \widetilde{U} = \bigoplus_{i=1}^{m} \widetilde{U}^{(j)}.$$

Note that $\widetilde{G}(\widetilde{U}^{(j)}) \subset \widetilde{U}^{(j)}$ and $\widetilde{G}$ acts faithfully on $\widetilde{U}$. From Theorem 1.5 follows that $K(\widetilde{G})$ is rational over $K(\widetilde{U})^{\widetilde{G}}$. We have

$$K(\widetilde{U})^{\widetilde{G}} = (K(\widetilde{U})^N)^G = (K(\widetilde{U}^{(1)})^N)^G \cdots (K(\widetilde{U}^{(m)})^N)^G.$$

Note that $(K(\widetilde{U}^{(j)})^N)^G$ is rational over $K(G)$, as we have shown in the proof of Theorem 1.3. Thus we obtain that $K(\widetilde{U})^{\widetilde{G}}$ is rational over the K-free compositum $K(G) \cdots K(G)$, which in turn is rational over K. We have done it.

REFERENCES

[1] Swan R. Noether's problem in Galois theory. In: Emmy Noether in Bryn Mawr (edited by B. Srinivasan and J. Sally), Springer-Verlag, Berlin, 1983.

[2] Saltman D. Adv. Math., **43**, 1982, 250–283.

[3] Garibaldi S., A. Merkurjev, J.-P. Serre. Cohomological invariants in Galois cohomology, AMS Univ. Lecture Series vol. **28**, Amer. Math. Soc., Providence, 2003.

[4] Kang M., B. Plans. Proc. Amer. Math. Soc., **137**, 2009, 1867–1874.

[5] Hajja S., M. Kang. J. Algebra, **177**, 1995, 511–535.

Faculty of Mathematics and Informatics
Episkop Konstantin Preslavski Shumen University
115, Universitetska Str.
9700 Shumen, Bulgaria
e-mail: ivo_michailov@yahoo.com