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THE RATIONALITY PROBLEM FOR THREE- AND FOUR-DIMENSIONAL PERMUTATIONAL GROUP ACTIONS

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Assume that K is a field, containing the full group of 4th roots of unity μ_4, and char $K \neq 2, 3$. Let G be a finite non-abelian subgroup of $\mathrm{GL}_n(K)$ for $n = 3$ or $n = 4$. The group G induces an action on $K(x_1, \ldots, x_n)$, the rational function field of n variables over K. Consider groups represented by matrices such that in each row and column there is exactly one element from μ_4 and all other elements are 0. With the aid of GAP [3] we find that there are precisely 230 such non-abelian groups in $\mathrm{SL}_4(K)$ and 33 in $\mathrm{GL}_3(K)$, up to conjugacy. We show that the fixed subfield $K(x_1, \ldots, x_n)^G$ is rational (i.e. purely transcendental) over K for every such group G. We also give a positive answer to the Noether's problem for several families of groups of order $m = 2^a \cdot 3^b$, where $a \geq 2$ and $b = 0, 1$.

Keywords: Noether's problem; rationality problem.

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1. Permutational Actions

Let us recall first the definition of monomial actions (see e.g. [7]). Let K be a field of char $K \neq 2$ and $K(x_1, \ldots, x_n)$ the rational function field over K with n variables $x_1, \ldots, x_n$. Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$ acting on $K(x_1, \ldots, x_n)$ by

$$\sigma(x_j) = c_j(\sigma) \prod_{i=1}^{n} x_i^{a_{i,j}}, \quad \sigma = (a_{i,j})_{1 \leq i,j \leq n} \in G, \quad c_i(\sigma) \in K^{\mathrm{x}}, \quad \text{for } 1 \leq j \leq n, \tag{1.1}$$

where $K^{\mathrm{x}} = K \backslash \{0\}$. We call this action of G *monomial*. If $c_j(\sigma) = 1$ for any $\sigma \in G$ and any $1 \leq j \leq n$ then the action of G is called *purely monomial*.

Let $m = 2^a \cdot 3^b$ for some $a \geq 2$ and $b \geq 0$. Assume that K is a field, containing the full group of mth roots of unity $\mu_m = \langle \zeta_m \rangle$, and $\operatorname{char} K \neq 2, 3$. Let $K(x_1, \ldots, x_n)$ be the rational function field over K with n variables $x_1, \ldots, x_n$. Let G be a finite subgroup of $\mathrm{GL}(n, K)$ acting on $K(x_1, \ldots, x_n)$ by

$$\sigma(x_i) = c_i(\sigma) \cdot x_{ji}, \quad \sigma \in G, \quad c_i(\sigma) \in \mu_m, \quad \text{for } 1 \leq i, \ j_i \leq n, \tag{1.2}$$

where $j_i \neq j_k$ for $i \neq k$.

It is easily verified that this action is well-defined. Indeed, if σ and τ are defined by equations such as (1.2), then the action of $\sigma\tau^{-1}$ is again of the same type. We call this action of G *permutational*. Each automorphism from G then is represented by a matrix of size n, such that in each row and column there is exactly one element from μ_m and all other elements are 0.

Let us now recall some well-known results.

Theorem 1.1 ([6, Theorem 1]). *Let G be a finite group acting on $L(x_1, \ldots, x_m)$, the rational function field of m variables over a field L such that*

(i) *for any $\sigma \in G, \sigma(L) \subset L$;*
(ii) *the restriction of the action of G to L is faithful;*
(iii) *for any $\sigma \in G$,*

$$\begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_m) \end{pmatrix} = A(\sigma) \begin{pmatrix} (x_1) \\ \vdots \\ (x_m) \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in \mathrm{GL}_m(L)$ and $B(\sigma)$ is $m \times 1$ matrix over L. Then there exist $z_1, \ldots, z_m \in L(x_1, \ldots, x_m)$ so that $L(x_1, \ldots, x_m)^G = L^G(z_1, \ldots, z_m)$ and $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq m$.

Theorem 1.2 ([1, Theorem 3.1]). *Let G be a finite group acting on $L(x)$, the rational function field of one variable over a field L. Assume that, for any $\sigma \in G, \sigma(L) \subset L$ and $\sigma(x) = a_\sigma x + b_\sigma$ for any $a_\sigma, b_\sigma \in L$ with $a_\sigma \neq 0$. Then $L(x)^G = L^G(z)$ for some $z \in L[x]$.*

Theorem 1.3 ([4, 5]). *Let K be any field and G a finite subgroup of $\mathrm{GL}(2, \mathbb{Z})$. Then the fixed field $K(x, y)^G$ under monomial action of G is rational over K.*

Next, we are going to prove several results, which will be applied later for the Noether's problem.

Theorem 1.4. *Let $m = 2^a \cdot 3^b$ for some $a \geq 2$ and $b \geq 0$. Assume that K is a field of* char $\neq 2, 3$ *that contains a primitive* mth *root of unity ζ, and let G be a group having a 3-dimensional monomial action* (1.1) *on $K(x, y, z)$ such that $c_j(\sigma) \in \mu_m$ for all $\sigma \in G, 1 \leq j \leq 3$. Then the fixed field $K(x, y, z)^G$ is rational over K.*

Proof. Since G has a 3-dimensional monomial action on $K(x, y, z)$, there exists a homomorphism $\rho : G \to \mathrm{GL}(3, \mathbb{Z})$. Denote $H = \mathrm{Ker}(\rho)$. Then G/H acts on

$K(x,y,z)^H$ via monomial actions. Moreover, the coefficients $c_j(\sigma)$ for all $\sigma \in G/H, 1 \le j \le 3$ are also in μ_m. Indeed, H consists of elements h such that $h(x) = \zeta^{\alpha h}x, h(y) = \zeta^{\beta_h}y, h(z) = \zeta^{\gamma_h}z$. Therefore, H is abelian and $K(x,y,z)^H = K(u,v,w)$ where u,v,w can be calculated as products of the kind $x^\alpha y^\beta z^\gamma$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$.

In this way, we may assume that G is a subgroup of $\mathrm{GL}(3,\mathbb{Z})$ such that $c_j(\sigma) \in \mu_m$ for all $\sigma \in G, 1 \le j \le 3$. It remains to apply [10, 7]. Namely, according to the results from the latter papers, the rationality is reduced to the conditions that either $-1 \in K^2$ or $[K(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) : K] \le 4$, where a_i's are some coefficients $c_j(\sigma) \in \mu_m$. Clearly, both conditions in our case are fulfilled. $\square$

Corollary 1.5. *Let $m = 2^a \cdot 3^b$ for some $a \ge 2$ and $b \ge 0$. Assume that K is a field of* char $\neq 2, 3$ *that contains a primitive* mth *root of unity ζ, and let G be a group having a 4-dimensional permutational action on $K(x_1, x_2, x_3, x_4)$ given by* (1.2)*. Then the fixed field $K(x_1, x_2, x_3, x_4)^G$ is rational over K.*

Proof. Define $y_1 = x_1, y_2 = x_2/x_1, y_3 = x_3/x_2, y_4 = x_3/x_2$. From Theorem 1.2 it follows that we can reduce the original rationality problem to the rationality problem for three dimensional monomial action on $K(y_2, y_3, y_4)$ such that $c_j(\sigma) \in \mu_m$ for all $\sigma \in G$. It remains to apply Theorem 1.4. $\square$

The following result is, of course, a straightforward application of Theorem 1.4, but we can give another proof which does not involve the results from [10, 7].

Corollary 1.6. *Let $m = 2^a \cdot 3^b$ for some $a \ge 2$ and $b \ge 0$. Assume that K is a field of* char $\neq 2, 3$ *that contains a primitive* mth *root of unity ζ, and let G be a finite subgroup of* $\mathrm{GL}(3, K)$ *with permutational action* (1.2) *on $K(x_1, x_2, x_3)$. Then $K(x_1, x_2, x_3)^G$ is rational over K.*

Proof. Define $y_1 = x_1, y_2 = x_2/x_1, y_3 = x_3/x_2$. From Theorem 1.2 it follows that we can reduce the original rationality problem to the rationality problem for two dimensional monomial action on $K(y_2, y_3)$ such that $c_j(\sigma) \in \mu_m$ for all $\sigma \in G$. We can apply now Theorem 1.3. $\square$

The purpose of this paper is to apply these corollaries in order to give a positive answer to the Noether's problem (see Sec. 4) for some groups of order $2^a \cdot 3^b$. In the next two sections we are going to describe in terms of matrix presentations all non-abelian groups having three- and four-dimensional permutational actions over fields containing a primitive 4th root of unity.

2. Subgroups with Three-Dimensional Permutational Action

Let K be a field, containing the full group of 4th roots of unity $\mu_4 = \langle i \rangle$, and let char $K \neq 2, 3$. Our goal in this section is to describe all non-abelian groups having a three dimensional permutational action (1.2), where $c_j(\sigma) \in \mu_4$ for all $\sigma \in G, 1 \le j \le 3$.

It is easily seen that the maximal subgroup of $\mathrm{GL}_3(K)$ with permutational action is of order $384 = 3 \cdot 2^7$. Denote it by G_{384}. With the help of [3] we obtain that G_{384} is generated by the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad iI = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ -1 & 0 & 0 \end{pmatrix}.$$

The matrices σ_1, σ_2 and iI generate a Sylow 2-subgroup G_{128}, which is isomorphic to $G_{32} \times C_4$, where G_{32} is generated by σ_1, σ_2, and C_4 is generated by iI. The group G_{384} is isomorphic to $G_{96} \times C_4$, where $G_{96} = G_{384} \cap \mathrm{SL}_3(K)$ is generated by σ_1, σ_2 and ρ. Since the Sylow 2-subgroups are conjugate and each 2-subgroup is included in some Sylow 2-subgroup, we need to find only the non-conjugate subgroups of G_{32} and G_{96}.

The program code from Appendix A gives the following four non-conjugate proper subgroups of $G_{32} = \langle \sigma_1, \sigma_2 \rangle$ having rank 2 (i.e. with minimal number of generators 2). Three groups of order 8: $G_{(8,1)} = \langle s_1, s_2 \rangle \cong D_4, G_{(8,2)} = \langle s_3, s_1 \rangle \cong D_4, G_{(8,3)} = \langle s_4, s_5 \rangle \cong Q_8$; one group of order 16: $G_{(16,1)} = \langle s_6, s_2 \rangle \cong M_{16}$ — the modular group, where

$$s_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$s_4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad s_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad s_6 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}.$$

There is only one non-abelian subgroup of G_{32} having rank 3: $G_{(16,2)} = \langle s_3, s_1, s_6 \rangle \cong DC$ — the central product of D_4 and C_4.

The program code from Appendix B gives the following four non-conjugate proper subgroups of $G_{96} = \langle \sigma_1, \sigma_2, \rho \rangle$ having rank ≤ 3 and order $3 \cdot 2^k$: $G_6 = \langle s_7, \rho \rangle \cong S_3$, $G_{12} = \langle s_2, \rho \rangle \cong A_4, G_{24} = \langle s_1, s_2, \rho \rangle \cong S_4$, and a group of order 48 — $G_{48} = \langle s_8, s_2, \rho \rangle$, where

$$s_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$s_8 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ -1 & 0 & 0 \end{pmatrix}.$$

Additional verification shows that there is not any non-abelian subgroup of G_{96} having rank 4 and order $3 \cdot 2^k$. In this way we have a total of 11 non-abelian and non-conjugate subgroups of G_{96}. If G is such a group then $G \times \langle -I \rangle$ and $G \times \langle iI \rangle$ are also non-abelian subgroups of G_{384}, where I is the identity matrix of size 3. Therefore, we have precisely 33 non-abelian and non-conjugate subgroups of $G_{384} \cong G_{96} \times \langle iI \rangle$.

3. Subgroups with Four-Dimensional Permutational Action

Let K be a field, containing the full group of 4th roots of unity $\mu_4 = \langle i \rangle$, and let char $K \neq 2, 3$. Our goal in this section is to describe all non-abelian subgroups in $\mathrm{SL}_4(K)$ with a permutational action, up to conjugacy in the maximal subgroup with permutational action in $\mathrm{GL}_4(K)$.

It is easily verified that the maximal subgroup of $\mathrm{GL}(4, K)$ with permutational action is of order $6144 = 3 \cdot 2048$. Denote it by $G_{3 \cdot 2048}$. It is generated by the matrices $m_1, \ldots, m_5, \rho$ and iI_4, where

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$m_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad m_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The maximal subgroup of $\mathrm{SL}(4, K)$ with permutational action is of order $1536 = 3 \cdot 512$. Denote it by $G_{3 \cdot 512}$. It is generated by the matrices $m_1, \ldots, m_5, \rho$. The matrices $m_j, 1 \leq j \leq 5$, generate a Sylow 2-subgroup of $G_{3 \cdot 512}$. Denote it by G_{512}.

Remark. The generating sets, given here are not minimal. Table C.1 in Appendix C shows that $g[108] \cong G_{512}$ has three generators.

Since the Sylow 2-subgroups are conjugate, we can reduce our computations to the description of a maximal family of pairwise non-conjugate 2-subgroups of G_{512}, without loss of generality. Moreover, we will concentrate on the non-abelian subgroups of $G_{3 \cdot 512}$, since for the abelian 2-groups we can apply [8]. (Note that the exponent of G_{512} is 8.) For the groups of order 3 we can apply [9]. We explain now the main steps of a GAP code which yields such a family of subgroups.

(1) Make a list $s[i]$ of all different non-abelian subgroups of G_{512}, generated by two elements. We obtain 919 such groups.
(2) For $i = 1$ to 918 and for $x \in G_{512}$, if $xs[i]x^{-1} = s[j]$ for $j = i + 1$ to 919 then remove every such group $s[j]$ from the list. Thus we obtain a list $h[i]$ with 179 groups which are pairwise non-conjugate in G_{512}.

(3) From the list $h[i]$ take the maximal number of groups which are pairwise non-conjugate in $G_{3\cdot 2048}$ Thus we obtain a list $g[i]$ with 69 groups, which is given in Table C.1 in Appendix C. We write each generator as a linear combination of the matrices $E_{i,j}$ having 1 (the only nonzero element) on the intersection of ith row and jth column.
(4) For each group $h[i], i = 1, \dots, 179$, and $x \in G_{512}$ make all subgroups $\langle h[i], x\rangle$ with exactly 3 generators. Thus we obtain another 604 groups which are pairwise non-conjugate in G_{512}. Add them to the list $s[i]$ of all different non-abelian subgroups of G_{512}, generated by two elements. From the list $s[i], i = 920, \dots, 1523$, take all pairwise non-conjugate subgroups in $G_{3\cdot 2048}$ and add them to the list $g[i]$. Thus we obtain 97 groups $g[i]$ of rank 3, $i = 70, \dots, 166$.
(5) For each group $s[i], i = 920, \dots, 1523$ and $x \in G_{512}$ make all subgroups $\langle s[i], x\rangle$ with exactly 4 generators. Verify that the new groups are not conjugate to any $s[i], i = 920, \dots, 1523$. Thus we obtain another 130 groups which are pairwise non-conjugate in G_{512}. Add them to the list $s[i]$. Take all pairwise non-conjugate in $G_{3\cdot 2048}$ and add them to the list $g[i]$. Thus we obtain 25 groups $g[i]$ of rank $4, i = 167, \dots, 191$.
(6) Similarly to the previous steps, we obtain that there are only 2 groups of rank 5, which are given in Table C.1 as well. There are no groups of rank 6.

Regarding the subgroups of order $3 \cdot 2^k$, we need to consider only the groups $\langle G_1, \rho\rangle$, where $G_1 \le G_{512}$. Indeed, let $G \le \mathrm{SL}_4(K)$ be a group of order $3 \cdot 2^k$. Then G contains a Sylow 2-subgroup G_1 and $G = \langle G_1, \rho_1\rangle$, where ρ_1 is of order 3. We may assume also that $G_1 \le G_{512}$, by taking a conjugate of G, if necessary. The Sylow 3-subgroups are conjugate in $G_{3.512}$, so $\rho_1 = g\rho g^{-1}$ for some $g \in G_{3.512}$. Since $g = h\rho^s$ for $h \in G_{512}, 0 \le s \le 2$, we have $\rho_1 = g\rho g^{-1} = h\rho h^{-1}$. Therefore, $h^{-1}Gh = \langle h^{-1}G_{-1}h, \rho\rangle$, where $h^{-1}G_1h \le G_{512}$.

With the aid of a GAP code resembling Steps (1)–(3), we obtain all non-abelian and pairwise non-conjugate subgroups of order $3 \cdot 2^k$ of rank ≤ 3. They are 37 and are given in Table C.2. There are no groups of rank 4.

The time needed for the first step of the algorithm is approximately 8 hours on a CPU 2800 GHz. We have to point out that we can not generate all subgroups with 3 and more generators by brute force, since the time will increase significantly (approximately $500^{k-2} \cdot 8$ hours for the subgroups with $k \ge 3$ generators). Instead, our algorithm requires max 24 hours for each step.

The following matrices give important information for the structure of the groups displayed in Tables C.1 and C.2. Denote $\mathcal{M} = \{c_j \,|\, j = 0, \dots, 7\} \cong C_2 \times C_2 \times C_2$, where

$$c_0 = I_4, \quad c_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$c_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad c_7 = -I_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $\mathcal{M}$ is normal in $G_{3\cdot 512}$, we get that $G \cap \mathcal{M}$ is normal in G for any subgroup G of $G_{3\cdot 512}$. This allows us to make an appropriate base change so that we "descend" from the initial functional field to the fixed subfield of $G \cap \mathcal{M}$. In this way one can show explicitly the rationality for each group.

4. Applications to Noether's Problem

Let K be a field and G be a finite group. Let G act on the rational function field $K(x(g) : g \in G)$ by K automorphisms defined by $g \cdot x(h) = x(gh)$ for any $g, h \in G$. Denote by $K(G)$ the fixed field $K(x(g) : g \in G)^G = \{f \in K(x(g) : g \in G) \,|\, \sigma \cdot f = f, \forall \sigma \in G\}$. Noether's problem then asks whether $K(G)$ is rational (= purely transcendental) over K.

We are going to give an affirmative answer to the Noether's problem for all groups described in Sec. 2, Tables C.1 and C.2.

Let K be a field, containing the full group of 4th roots of unity $\mu_4 = \langle i \rangle$, and let char $K \neq 2, 3$. Let G be isomorphic to any of the described groups. Since char $K \neq 2, 3$, the group algebra $K[G]$ is semi-simple. Hence any three- or four-dimensional faithful representation can be embedded into the regular representation whose dual space is $V_{\text{reg}} = \oplus_{g \in G} K \cdot x(g)$, where G acts on V_{reg} by $h \cdot x(g) = x(hg)$ for any $h, g \in G$. Applying Theorem 1.1, we find that $K(x(g) : g \in G)^G$ is rational over $K(x_1, x_2, x_3)^G$ or, respectively, over $K(x_1, x_2, x_3, x_4)^G$, which in turn is rational over K, as we have shown in Corollaries 1.5 and 1.6.

The question naturally arises as to whether we can find more groups of orders 2^{m+3} and $3 \cdot 2^{m+3}$ for $m \geq 0$, which posses four dimensional permutational action over fields containing suitable roots of unity. In what follows, we are going to discuss several families of such groups in terms of generators and relations.

(I) Let G be a non-abelian group of order $8n$, having a cyclic subgroup of index 2, where $n = 3^k \cdot 2^m$ for some $m \geq 0, k = 0, 1$. Then G is generated by two elements σ and τ such that $\sigma^{4n} = 1, \tau^2 = \sigma^a$ and $\tau\sigma = \sigma^r \tau$, where $a, r \in \mathbb{Z}$ should be subject to some restriction. For example, from $\tau^2 \sigma \tau^{-3} = \sigma^{r^2}$ it follows that r must be a solution to the congruence

$$x^2 \equiv 1 \pmod{4n}. \tag{4.1}$$

Therefore $r = 1 + 2s$, where $s \not\equiv 0 \pmod{2n}$ is an integer such that

$$s(s+1) \equiv 0 \pmod{n}. \tag{4.2}$$

In order to find all different solutions to (4.1), we need to consider the solutions to (4.2) as classes mod $2n$ (rather than mod n). Clearly, trivial solutions to (4.2) are $s = -1, n-1$ and $s = n$, whence $r = -1, \pm 1 + 2n$ are solutions to (4.1).

Now, if $k = 0$, i.e., G is a 2-group, the mentioned trivial solutions are the only solutions. In this case it is well-known that there exist exactly four non-isomorphic groups — the dihedral, semidihedral, quaternion and the modular group.

If $k = 1$, there is a non-trivial solution to (4.2). Since one of the numbers $2^m - 1$ or $2^m + 1$ is divisible by 3, we can put $s = 2^m - 1$ or $s = 2^m$, respectively.

Assume now that the field K contains a primitive $2n$th root of unity ζ. Consider the group algebra $K[G]$ and define $X = \sum_i \zeta^{-i} x(\sigma^{2i})$. Then $\sigma^2(X) = \zeta X$, and define $x_1 = X, x_2 = \sigma X, x_3 = \tau X, x_4 = \tau\sigma X$. Clearly, we obtain a faithful permutational action, so from Corollary 1.6 it follows that $K(x_1, x_2, x_3, x_4)^G$ is rational over K. Therefore, the Noether's problem has an affirmative answer.

(II) Let G be a non-abelian group of order $16n$, having a cyclic subgroup of index 4, where $n = 3^k \cdot 2^m$ for some $m \geq 0, k = 0, 1$. Choose an element σ of order $4n$. Assume additionally that $C_G(\sigma) \neq \langle \sigma \rangle$. Then G possesses an abelian subgroup $H = \langle \sigma, \tau \rangle \cong C_{4n} \times C_2$ and has an element λ such that $G = \langle H, \lambda \rangle$ and $\lambda^2 \in H$. We must have $\lambda\sigma\lambda^{-1} = \sigma^i \tau^k ((2n, i) = 1, 0 \leq k \leq 1)$ and $\lambda\tau\lambda^{-1} = \sigma^{2nl}\tau^j (0 \leq l, j \leq 1)$.

Assume again that the field K contains a primitive $2n$th root of unity ζ. Consider the group algebra $K[G]$ and define $X = \sum_i \zeta^{-i}[x(\sigma^{2i}) + x(\sigma^{2i}\tau)]$. Then $\sigma^2(X) = \zeta X$ and $\tau X = X$. Define $x_1 = X, x_2 = \sigma X, x_3 = \lambda X, x_4 = \lambda\sigma X$. In this way, we obtain a faithful permutational action, so again from Corollary 1.6 it follows that $K(x_1, x_2, x_3, x_4)^G$ is rational over K.

(III) We will consider four types of 2-groups which are discussed in [2]:

$$\begin{aligned} D(a,b) = \langle \sigma, \tau, \lambda : \sigma^{2^a} = \tau^{2^b} = \lambda^2 = 1,\ \lambda\sigma\lambda^{-1} = \sigma^{-1}, \\ \lambda\tau\lambda^{-1} = \tau^{-1}, \sigma\tau = \tau\sigma \rangle, \end{aligned}$$

$$\begin{aligned} Q(a,b) = \langle \sigma, \tau, \lambda : \sigma^{2^{a-1}} = \lambda^2, \tau^{2^b} = \lambda^4 = 1,\ \lambda\sigma\lambda^{-1} = \sigma^{-1}, \\ \lambda\tau\lambda^{-1} = \tau^{-1}, \sigma\tau = \tau\sigma \rangle, \end{aligned}$$

$$\begin{aligned} SD(a,b) = \langle \sigma, \tau, \lambda : \sigma^{2^a} = \tau^{2^b} = \lambda^2 = 1, \lambda\sigma\lambda^{-1} = \sigma^{-1+2^{a-1}}, \\ \lambda\tau\lambda^{-1} = \tau^{-1}, \sigma\tau = \tau\sigma \rangle, \end{aligned}$$

$$\begin{aligned} DS(a,b) = \langle \sigma, \tau, \lambda : \sigma^{2^a} = \tau^{2^b} = \lambda^2 = 1,\ \lambda\sigma\lambda^{-1} = \sigma^{-1}\tau^{2^{b-1}}, \\ \lambda\tau\lambda^{-1} = \tau^{-1}, \sigma\tau = \tau\sigma \rangle, \end{aligned}$$

where $a \geq 1(a \geq 2$ in the SD-case) and $b \geq 1$. Clearly, each of these presentations yields a group of order 2^{a+b+1}.

Let G be isomorphic to any of the above groups. Assume that the field K contains a primitive 2^ath root of unity ζ and a primitive 2^bth root of unity ξ. Define in the group algebra $K[G]$ the elements

$$X = \sum_{i=0}^{2^a-1} \zeta^{-i}[x(\sigma^i) + x(\sigma^i\tau) + \cdots + x(\sigma^i\tau^{2^b-1})],$$

$$Y = \sum_{i=0}^{2^b-1} \xi^{-i}[x(\sigma^i) + x(\tau^i\sigma) + \cdots + x(\tau^i\sigma^{2^a-1})].$$

We have $\sigma X = \zeta X, \sigma Y = Y, \tau X = X, \tau Y = \xi Y$. Define $x_1 = X, x_2 = Y, x_3 = \lambda X, x_4 = \lambda Y$. It is easily seen now that we obtain a faithful four-dimensional permutational action on $K(x_1, x_2, x_3, x_4)$, whence the answer to the Noether's problem is affirmative.

Appendix A. GAP Code Providing All Non-Abelian and Non-Conjugate Subgroups of G_{32}

```
#The following code defines the group m=G_{32} and computes the
#different non-abelian subgroups s[t] with generators s1[t] and
#s2[t]

m1:=[[0,0,E(4)], [0,E(4),0], [1,0,0]];
m2:=[[0,0,E(4)], [0,1,0], [E(4),0,0]];
m:= Group(m1,m2); s:=[1..100]; s1:=[1..100]; s2:=[1..100];
a:=[1..32];
i:= 0;
for p in m do
 i:= i+1;
 a[i]:= p;
od;
r:= 1; e:=[[1,0,0], [0,1,0], [0,0,1]]; s[1]:= Group(e);
for l in
[1..32] do
 for j in [1..l] do
  k:= Group (a[l],a[j]);
  q:= 1;
  if IsAbelian(k) then
   q:= q*0;
  else
```

```
   for i in [1..r] do
    if k = s[i] then
     q:= q*0;
  else
   q:= q*1;
  fi;
 od;
fi;
if q=1 then
 r:= r+1;
 s[r]:= k;
 s1[r]:= a[l];
 s2[r]:= a[j];
  fi;
 od;
od;

#The following code gives the non-conjugate subgroups of rank 2

ss:=[1..100];
for u in [2..r] do
 ss[u]:= 1;
od;
for t in [2..r-1] do
 if ss[t]= 1 then
 for g in m do
  k:= Group(g*s1[t]*g^-1,g*s2[t]*g^-1);
  for u in [t+1..r] do
   if k = s[u] then
    ss[u]:= 0;
   fi;
  od;
 od;
 fi;
od;
x:= 0;
for u in [2..r] do
 x:= x+ss[u];
od;
Display("Number of non-conjugate groups of rank 2:");
Display(x);
for t in [2..r] do
```

```
 if ss[t]=1 then
  Print("#Size: ",Size(s[t]));
  Display("");
  Print("s1[",t,"]:=");
  Display("");
  Display(s1[t]);
  Display(";");
  Print("s2[",t,"]:=");
  Display("");
  Display(s2[t]);
  Display(";");
 fi;
od;

#The following code gives all non-abelian subgroups of rank 3

r0:= r; s3:= [1..100]; e:= [[1,0,0], [0,1,0], [0,0,1]];
 s[1]:= Group(e);
for i in [2..r] do
 s[i]:= Group(s1[i],s2[i]);
od;
for l in [1..32] do
 for j in [1..l] do
  for i in [1..j] do
   k:= Group(a[l],a[j],a[i]);
   q:= 1;
   if IsAbelian(k) then
    q:= 0;
   else
    for n in [1..r] do
     if k= s[n] then
      q:= q*0;
     else
      q:= q*1;
     fi;
    od;
   fi;
   if q=1 then
    r:= r+1;
    s[r]:= k;
    s1[r]:= a[l];
    s2[r]:= a[j];
```

```
    s3[r]:= a[i];
   fi;
  od;
 od;
od;
for t in [r0+1..r] do
  Print("#Size: ",Size(s[t]));
  Display("");
  Print("s1[",t,"]:=");
  Display("");
  Display(s1[t]);
  Display(";");
  Print("s2[",t,"]:=");
  Display("");
  Display(s2[t]);
  Display(";");
  Print("s3[",t,"]:=");
  Display("");
  Display(s3[t]);
  Display(";");
od;
```

Appendix B. GAP Code Providing All Non-Abelian Subgroups of G_{96} of Order $3 \cdot 2^k$

```
#The following code computes the different non-abelian subgroups
#s[t] of G_{96} with generators s1[t], s2[t], m3

m1:=[[0,0,E(4)], [0,E(4),0], [1,0,0]];
m2:=[[0,0,E(4)], [0,1,0],
[E(4),0,0]];
m3:=[[0,E(4),0], [0,0,E(4)], [-1,0,0]];
m:= Group(m1,m2);
s:=[1..100]; s1:=[1..100]; s2:=[1..100];
a:=[1..32]; i:=0;
for p in m do
 i:= i+1;
 a[i]:= p;
od;
r:= 1; e:=[[1,0,0], [0,1,0], [0,0,1]]; s[1]:= Group(e);
for l in
[1..32] do
 for j in [1..l] do
```

```
  k:= Group(a[l],a[j],m3);
  q:= 1;
  if IsAbelian(k) then
   q:= q*0;
  else
   for i in [1..r] do
    if k = s[i] then
     q:= q*0;
    else
     q:= q*1;
    fi;
   od;
  fi;
  if q = 1 then
   r:= r+1;
   s[r]:= k;
   s1[r]:= a[l];
   s2[r]:= a[j];
  fi;
 od;
od;
Print("m3:="); Display(""); Display(m3); Display(";");
 for t in [2..r] do
  Print("#Size: ",Size(s[t]));
  Display("") ;
  Print("s1 [",t,"]:=");
  Display("");
  Display(s1[t]);
  Display(";");
  Print("s2[",t,"]:=");
  Display("");
  Display(s2[t]);
  Display(";");
 od;
```

Appendix C. Tables of the Non-Abelian Subgroups of $\mathbf{SL}_4(\boldsymbol{K})$

Table C.1. Non-abelian and pairwise non-conjugate 2-subgroups in G_{512}.

$g[j]$	Generators	$9[j] \cap \mathcal{M}$	ord
$j=1$	$g_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	8
$j=2$	$g_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\langle c_4, c_5\rangle$	32
$j=3$	$g_1 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	8
$j=4$	$g_1 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	16
$j=5$	$g_1 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4}, g_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\langle c_4, c_5\rangle$	32

Table C.1. (*Continued*)

$g[j]$	Generators	$\mathfrak{g}[j]\cap\mathcal{M}$	ord
$j=6$	$g_1=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_4,c_5\rangle$	32
$j=7$	$g_1=-E_{1,1}-E_{2,3}+iE_{3,2}+iE_{4,4}, g_2=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}$	$\langle c_4,c_5\rangle$	16
$j=8$	$g_1=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$	$\langle c_4\rangle$	8
$j=9$	$g_1=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_4\rangle$	8
$j=10$	$g_1=-E_{1,1}-iE_{2,2}+iE_{3,3}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_4\rangle$	8
$j=11$	$g_1=-E_{1,1}-iE_{2,2}-iE_{3,3}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$	$\langle c_4\rangle$	8
$j=12$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-E_{2,2}+iE_{3,3}+iE_{4,4}$	$\langle c_6\rangle$	8
$j=13$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_7$	16
$j=14$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$	$\langle c_3,c_7\rangle$	32
$j=15$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}$	$\langle c_3,c_7\rangle$	128
$j=16$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$	$\langle c_2,c_7\rangle$	8
$j=17$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-iE_{3,3}-iE_{4,4}$	$\langle c_1,c_7\rangle$	16
$j=18$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-iE_{2,2}-E_{3,3}+iE_{4,4}$	$\langle c_2,c_7\rangle$	32
$j=19$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-iE_{2,2}+E_{3,3}-iE_{4,4}$	$\langle c_2,c_7\rangle$	32
$j=20$	$g_1=-E_{1,2}-E_{2,1}-iE_{3,4}+iE_{4,3}, g_2=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}$	$\langle c_6\rangle$	8
$j=21$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}$	$\mathcal{M}$	32
$j=22$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-E_{2,2}-iE_{3,3}+iE_{4,4}$	$\mathcal{M}$	256
$j=23$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}+E_{2,3}+E_{3,2}+E_{4,4}$	$\langle c_7\rangle$	16
$j=24$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=25$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=26$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}+E_{2,2}+E_{3,3}-E_{4,4}$	$\langle c_3,c_7\rangle$	16
$j=27$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=28$	$g_1=-E_{1,2}-E_{2,4}-E_{3,1}+E_{4,3}, g_2=-E_{1,1}-iE_{2,2}+iE_{3,3}-E_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=29$	$g_1=-E_{1,2}+E_{2,1}-E_{3,4}+E_{4,3}, g_2=-E_{1,1}-E_{2,2}-iE_{3,3}+iE_{4,4}$	$\langle c_1,c_7\rangle$	16
$j=30$	$g_1=-E_{1,2}-iE_{2,1}-E_{3,4}+iE_{4,3}, g_2=-E_{1,1}-E_{2,2}-iE_{3,3}+iE_{4,4}$	$\langle c_1,c_7\rangle$	32
$j=31$	$g_1=-E_{1,2}-iE_{2,1}-E_{3,4}+iE_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$	$\langle c_2,c_7\rangle$	16
$j=32$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-E_{2,1}+E_{3,4}+E_{4,3}$	$\langle c_7\rangle$	8
$j=33$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-E_{2,1}-iE_{3,4}+iE_{4,3}$	$\langle c_7\rangle$	16
$j=34$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-iE_{2,4}-iE_{3,1}-E_{4,3}$	$\langle c_3,c_7\rangle$	32
$j=35$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-iE_{2,4}+iE_{3,1}+E_{4,3}$	$\langle c_7\rangle$	16
$j=36$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}+E_{2,1}-iE_{3,4}-iE_{4,3}$	$\langle c_7\rangle$	16
$j=37$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-iE_{2,1}-E_{3,4}+iE_{4,3}$	$\langle c_2,c_7\rangle$	32
$j=38$	$g_1=-E_{1,3}-E_{2,4}+E_{3,1}+E_{4,2}, g_2=-E_{1,2}-iE_{2,4}-iE_{3,1}-E_{4,3}$	$\langle c_7\rangle$	16
$j=39$	$g_1=-E_{1,3}-E_{2,4}+E_{3,1}+E_{4,2}, g_2=-E_{1,2}+E_{2,1}+E_{3,4}-E_{4,3}$	$\langle c_7\rangle$	8
$j=40$	$g_1=-E_{1,3}-E_{2,4}+E_{3,1}+E_{4,2}, g_2=-E_{1,2}+E_{2,1}-iE_{3,4}-iE_{4,3}$	$\langle c_7\rangle$	16
$j=41$	$g_1=-E_{1,3}-E_{2,4}+E_{3,1}+E_{4,2}, g_2=-E_{1,2}-iE_{2,1}-E_{3,4}+iE_{4,3}$	$\langle c_2,c_7\rangle$	32
$j=42$	$g_1=-E_{1,3}-E_{2,4}-iE_{3,1}+iE_{4,2}, g_2=-E_{1,2}-iE_{2,1}-iE_{3,4}+E_{4,3}$	$\mathcal{M}$	64
$j=43$	$g_1=-E_{1,3}-iE_{2,1}-E_{3,4}-iE_{4,2}, g_2=-E_{1,2}-E_{2,4}+iE_{3,1}+iE_{4,3}$	$\langle c_3,c_7\rangle$	32
$j=44$	$g_1=-E_{1,3}-iE_{2,1}-E_{3,4}-iE_{4,2}, g_2=-E_{1,2}+E_{2,4}+iE_{3,1}-iE_{4,3}$	$\langle c_7\rangle$	16
$j=45$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_3\rangle$	8
$j=46$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=47$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$	$\langle c_3,c_7\rangle$	16
$j=48$	$g_1=-E_{1,4}-E_{2,2}+E_{3,3}-E_{4,1}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_7\rangle$	8
$j=49$	$g_1=-E_{1,4}-E_{2,2}+E_{3,3}-E_{4,1}, g_2=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=50$	$g_1=-E_{1,4}-E_{2,2}+E_{3,3}-E_{4,1}, g_2=-E_{1,1}-iE_{2,2}+E_{3,3}-iE_{4,4}$	$\langle c_1,c_3\rangle$	32
$j=51$	$g_1=-E_{1,4}-E_{2,2}+E_{3,3}-E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$	$\langle c_3,c_7\rangle$	16
$j=52$	$g_1=-E_{1,4}-E_{2,2}-iE_{3,3}-iE_{4,1}, g_2=-E_{1,1}-E_{2,3}-iE_{3,2}-iE_{4,4}$	$\langle c_3,c_7\rangle$	64
$j=53$	$g_1=-E_{1,4}-E_{2,2}-iE_{3,3}-iE_{4,1}, g_2=-E_{1,1}-iE_{2,2}-E_{3,3}+iE_{4,4}$	$\mathcal{M}$	64
$j=54$	$g_1=-E_{1,4}-E_{2,2}-iE_{3,3}-iE_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$	$\langle c_3,c_7\rangle$	32
$j=55$	$g_1=-E_{1,4}-E_{2,3}-iE_{3,2}-iE_{4,1}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$	$\langle c_3\rangle$	8
$j=56$	$g_1=-E_{1,4}-E_{2,3}-E_{3,2}-E_{4,1}, g_2=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}$	$\langle c_7\rangle$	8
$j=57$	$g_1=-E_{1,4}-E_{2,3}+E_{3,2}+E_{4,1}, g_2=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}$	$\langle c_3,c_7\rangle$	16

Table C.1. (*Continued*)

$g[j]$	Generators	$9[j] \cap \mathcal{M}$	ord
$j = 58$	$g_1 = -E_{1,4} - E_{2,3} + E_{3,2} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}$	$\langle c_3, c_7\rangle$	16
$j = 59$	$g_1 = -E_{1,4} + E_{2,2} - iE_{3,3} + iE_{4,1}, g_2 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4}$	$\langle c_2, c_3\rangle$	16
$j = 60$	$g_1 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	16
$j = 61$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$	$\langle c_3, c_7\rangle$	16
$j = 62$	$g_1 = -iE_{1,4} - E_{2,2} + E_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$	$\langle c_3, c_7\rangle$	16
$j = 63$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1}$	$\langle c_7\rangle$	8
$j = 64$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$	$\langle c_3, c_7\rangle$	16
$j = 65$	$g_1 = -iE_{1,4} - E_{2,3} + E_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,3} - iE_{3,2} + E_{4,1}$	$\langle c_7\rangle$	8
$j = 66$	$g_1 = -iE_{1,4} - E_{2,3} + E_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$	$\langle c_7\rangle$	8
$j = 67$	$g_1 = -iE_{1,4} + E_{2,2} + E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} + E_{2,2} + E_{3,3} + E_{4,1}$	$\langle c_3\rangle$	8
$j = 68$	$g_1 = -iE_{1,4} - iE_{2,2} - iE_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$	$\langle c_3, c_7\rangle$	16
$j = 69$	$g_1 = -iE_{1,4} - iE_{2,2} + iE_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$	$\langle c_3, c_7\rangle$	16
$j = 70$	$g_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	16
$j = 71$	$g_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	64
$j = 72$	$g_1 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	32
$j = 73$	$g_1 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}, g_2 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_3, c_7\rangle$	16
$j = 74$	$g_1 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_3, c_7\rangle$	16
$j = 75$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_1, c_7\rangle$	16
$j = 76$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_3, c_7\rangle$	16
$j = 77$	$g_1 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}, g_2 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	16
$j = 78$	$g_1 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	16
$j = 79$	$g_1 = -E_{1,1} - iE_{2,2} + iE_{3,3} - E_{4,4}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2\rangle$	16
$j = 80$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 81$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	256
$j = 82$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	16
$j = 83$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - iE_{2,2} - E_{3,3} + iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 84$	$g_1 = -E_{1,2} - E_{2,4} - E_{3,1} + E_{4,3}, g_2 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	128
$j = 85$	$g_1 = -E_{1,2} - E_{2,4} - E_{3,1} + E_{4,3}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	128
$j = 86$	$g_1 = -E_{1,2} - iE_{2,1} - E_{3,4} + iE_{4,3}, g_2 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 87$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - E_{2,1} + E_{3,4} + E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	16
$j = 88$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - E_{2,1} - iE_{3,4} + iE_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	32
$j = 89$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - iE_{2,4} - iE_{3,1} - E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64

Table C.1. (*Continued*)

$g[j]$	Generators	$9[j] \cap \mathcal{M}$	ord
$j = 90$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} + E_{2,1} + E_{3,4} - E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	16
$j = 91$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} + E_{2,1} - iE_{3,4} - iE_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	32
$j = 92$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - iE_{2,1} - E_{3,4} + iE_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 93$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - iE_{2,1} - iE_{3,4} + E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	32
$j = 94$	$g_1 = -E_{1,3} - iE_{2,1} - E_{3,4} - iE_{4,2}, g_2 = -E_{1,2} - E_{2,4} + iE_{3,1} + iE_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 95$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,4} + E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 96$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	128
$j = 97$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 98$	$g_1 = -E_{1,4} - E_{2,2} - iE_{3,3} - iE_{4,1}, g_2 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	128
$j = 99$	$g_1 = -E_{1,4} - E_{2,2} - iE_{3,3} - iE_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 100$	$g_1 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 101$	$g_1 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 102$	$g_1 = -E_{1,4} - E_{2,3} - iE_{3,2} + iE_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	64
$j = 103$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 104$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	16
$j = 105$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 106$	$g_1 = -iE_{1,4} - iE_{2,3} - iE_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - E_{2,3} + E_{3,2} + E_{4,1},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_7\rangle$	16
$j = 107$	$g_1 = -iE_{1,4} - iE_{2,2} - iE_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1},$ $g_3 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	32
$j = 108$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	512
$j = 109$	$g_1 = -E_{1,4} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,4} - iE_{2,2} - E_{3,3} + iE_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	128
$j = 110$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - E_{2,1} + E_{3,4} + E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\langle c_1, c_7\rangle$	64
$j = 111$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} + E_{2,1} + E_{3,4} - E_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\langle c_1, c_7\rangle$	64
$j = 112$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - iE_{2,1} - E_{3,4} + iE_{4,3},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	128
$j = 113$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	256
$j = 114$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	128
$j = 115$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4},$ $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7\rangle$	128

Table C.1. (*Continued*)

$g[j]$	Generators	$9[j] \cap \mathcal{M}$	ord
$j = 116$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	128
$j = 117$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} + E_{2,2} + E_{3,3} - E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 118$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 119$	$g_1 = -E_{1,4} - E_{2,2} - E_{3,3} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 120$	$g_1 = -E_{1,4} - E_{2,2} + E_{3,3} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 121$	$g_1 = -E_{1,4} - E_{2,2} + E_{3,3} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 122$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 123$	$g_1 = -iE_{1,4} - E_{2,2} + E_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 124$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 125$	$g_1 = -iE_{1,4} - E_{2,3} + E_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	16
$j = 126$	$g_1 = -iE_{1,4} - iE_{2,3} - iE_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 127$	$g_1 = -iE_{1,4} + E_{2,2} + E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} + E_{2,2} + E_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3 \rangle$	16
$j = 128$	$g_1 = -iE_{1,4} - iE_{2,2} - iE_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 129$	$g_1 = -iE_{1,4} - iE_{2,2} + iE_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 130$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} + E_{3,1} - iE_{4,2}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	32
$j = 131$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} + iE_{2,4} + E_{3,1} + iE_{4,2}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	64
$j = 132$	$g_1 = -iE_{1,3} - E_{2,4} + iE_{3,1} - E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} - E_{3,1} + iE_{4,3}$, $g_3 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	32
$j = 133$	$g_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, g_2 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_4, c_7 \rangle$	128
$j = 134$	$g_1 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} + iE_{3,3} - E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_4 \rangle$	16
$j = 135$	$g_1 = -E_{1,4} - E_{2,3} + E_{3,2} + E_{4,1}, g_2 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 136$	$g_1 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1}, g_2 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_4 \rangle$	16
$j = 137$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 138$	$g_1 = -iE_{1,4} - iE_{2,3} - iE_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 139$	$g_1 = -iE_{1,4} - E_{2,3} + iE_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 140$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} + E_{3,1} - iE_{4,2}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	64
$j = 141$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} + iE_{2,4} + E_{3,1} + iE_{4,2}$, $g_3 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}$	$\langle c_3, c_7 \rangle$	64

Table C.1. (*Continued*)

$g[j]$	Generators	$\mathfrak{g}[j] \cap \mathcal{M}$	ord
$j = 142$	$g_1 = -E_{1,4} - E_{2,2} + E_{3,3} - E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	16
$j = 143$	$g_1 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1}, g_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4},$ $g_3 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle c_3, c_7 \rangle$	16
$j = 144$	$g_1 = -iE_{1,4} - E_{2,3} + E_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	16
$j = 145$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} + E_{2,1} + E_{3,4} - E_{4,3},$ $g_3 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	32
$j = 146$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} + E_{3,1} - iE_{4,2},$ $g_3 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4}$	$\langle c_2, c_7 \rangle$	16
$j = 147$	$g_1 = -iE_{1,3} - E_{2,4} + iE_{3,1} - E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} - E_{3,1} + iE_{4,2},$ $g_3 = -E_{1,1} + E_{2,2} - E_{3,3} + E_{4,4}$	$\langle c_2, c_7 \rangle$	16
$j = 148$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - iE_{2,4} + iE_{3,1} + E_{4,3},$ $g_3 = -E_{1,1} + E_{2,2} + E_{3,3} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 149$	$g_1 = -E_{1,3} - iE_{2,1} - E_{3,4} - iE_{4,2}, g_2 = -E_{1,2} + E_{2,4} + iE_{3,1} - iE_{4,3},$ $g_3 = -E_{1,1} + E_{2,2} + E_{3,3} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 150$	$g_1 = -iE_{1,4} - E_{2,3} + E_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} + E_{2,2} + E_{3,3} - E_{4,4}$	$\langle c_3, c_7 \rangle$	16
$j = 151$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - E_{2,1} + E_{3,4} + E_{4,3},$ $g_3 = -E_{1,1} + E_{2,2} - iE_{3,3} - iE_{4,4}$	$\langle c_1, c_7 \rangle$	64
$j = 152$	$g_1 = -E_{1,3} - E_{2,4} - E_{3,1} - E_{4,2}, g_2 = -E_{1,2} - E_{2,1} - iE_{3,4} + iE_{4,3},$ $g_3 = -E_{1,1} + E_{2,2} - iE_{3,3} - iE_{4,4}$	$\langle c_1, c_7 \rangle$	64
$j = 153$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} + E_{3,1} - iE_{4,2},$ $g_3 = -E_{1,1} - iE_{2,2} - E_{3,3} + iE_{4,4}$	$\langle c_2, c_7 \rangle$	32
$j = 154$	$g_1 = -iE_{1,3} - E_{2,4} + iE_{3,1} - E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} - E_{3,1} + iE_{4,2},$ $g_3 = -E_{1,1} - iE_{2,2} - E_{3,3} + iE_{4,4}$	$\langle c_2, c_7 \rangle$	32
$j = 155$	$g_1 = -iE_{1,3} - E_{2,4} - iE_{3,1} + E_{4,2}, g_2 = -E_{1,3} - iE_{2,4} + E_{3,1} - iE_{4,2},$ $g_3 = -E_{1,1} - iE_{2,2} + E_{3,3} - iE_{4,4}$	$\langle c_2, c_7 \rangle$	32
$j = 156$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 157$	$g_1 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 158$	$g_1 = -iE_{1,4} + E_{2,2} + E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} + E_{2,2} + E_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 159$	$g_1 = -iE_{1,4} - iE_{2,2} + iE_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 160$	$g_1 = -iE_{1,4} - E_{2,2} - E_{3,3} - iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} + iE_{3,3} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 161$	$g_1 = -iE_{1,4} - E_{2,2} + E_{3,3} + iE_{4,1}, g_2 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1},$ $g_3 = -E_{1,1} - iE_{2,2} + iE_{3,3} - E_{4,4}$	$\langle c_3, c_7 \rangle$	32
$j = 162$	$g_1 = -iE_{1,4} - iE_{2,3} - iE_{3,2} - iE_{4,1}, g_2 = -E_{1,4} - E_{2,3} + E_{3,2} + E_{4,1},$ $g_3 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}$	$\langle c_7 \rangle$	16
$j = 163$	$g_1 = -iE_{1,4} - iE_{2,3} + iE_{3,2} + iE_{4,1}, g_2 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1},$ $g_3 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}$	$\langle c_7 \rangle$	16
$j = 164$	$g_1 = -iE_{1,3} - iE_{2,4} - iE_{3,1} - iE_{4,2}, g_2 = -E_{1,3} + E_{2,4} + E_{3,1} - E_{4,2},$ $g_3 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}$	$\langle c_7 \rangle$	16
$j = 165$	$g_1 = -E_{1,3} - iE_{2,4} - E_{3,1} + iE_{4,2}, g_2 = -E_{1,2} + E_{2,1} + iE_{3,4} + iE_{4,3},$ $g_3 = -E_{1,4} - iE_{2,3} - iE_{3,2} + E_{4,1}$	$\langle c_7 \rangle$	16
$j = 166$	$g_1 = -E_{1,4} - E_{2,3} - E_{3,2} - E_{4,1}, g_2 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4},$ $g_3 = -iE_{1,4} - E_{2,3} - E_{3,2} + iE_{4,1}$	$\langle c_3, c_7 \rangle$	32
$j = 167$	$g_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, g_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}, g_4 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}$	$\mathcal{M}$	32
$j = 168$	$g_1 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}, g_2 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4},$ $g_3 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}, g_4 = -E_{1,4} - E_{2,2} + E_{3,3} - E_{4,1}$	$\langle c_3, c_7 \rangle$	32

Table C.1. (*Continued*)

$g[j]$	Generators	$9[j]\cap\mathcal{M}$	ord
$j=169$	$g_1=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,3}, g_4=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}$	$\mathcal{M}$	256
$j=170$	$g_1=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}$	$\mathcal{M}$	64
$j=171$	$g_1=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,1}-E_{2,2}-E_{3,3}+iE_{4,4}$	$\mathcal{M}$	64
$j=172$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}-iE_{3,2}-iE_{4,1}$	$\mathcal{M}$	128
$j=173$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}$	$\mathcal{M}$	32
$j=174$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,3}-iE_{2,4}-E_{3,1}+iE_{4,2}$	$\mathcal{M}$	64
$j=175$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,2}-iE_{2,1}-iE_{3,4}-iE_{4,3}$	$\mathcal{M}$	32
$j=176$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-iE_{2,2}-E_{3,3}+iE_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}$	$\mathcal{M}$	128
$j=177$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-E_{2,1}+E_{3,4}+E_{4,3}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}-iE_{3,2}-iE_{4,1}$	$\langle c_1,c_7\rangle$	32
$j=178$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}-E_{2,1}+E_{3,4}+E_{4,3}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}+iE_{2,3}-iE_{3,2}+iE_{4,1}$	$\langle c_1,c_7\rangle$	32
$j=179$	$g_1=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}, g_2=-E_{1,2}+E_{2,1}+E_{3,4}-E_{4,3}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}-iE_{3,2}-iE_{4,1}$	$\langle c_1,c_7\rangle$	32
$j=180$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}-iE_{3,2}-iE_{4,1}$	$\mathcal{M}$	64
$j=181$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}-E_{3,2}+iE_{4,1}$	$\mathcal{M}$	64
$j=182$	$g_1=-E_{1,4}-E_{2,3}-E_{3,2}-E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}-E_{3,3}+E_{4,4}, g_4=-iE_{1,4}-E_{2,3}-E_{3,2}+iE_{4,1}$	$\mathcal{M}$	64
$j=183$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}-iE_{2,2}-E_{3,3}+iE_{4,4}$, $g_3=-E_{1,1}-E_{2,2}-iE_{3,3}+iE_{4,4}, g_4=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}$	$\mathcal{M}$	256
$j=184$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}+E_{3,3}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-iE_{2,3}+iE_{3,2}+iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=185$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-E_{2,2}-E_{3,3}-iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=186$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-E_{2,2}+E_{3,3}+iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=187$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-E_{2,2}-E_{3,3}-iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=188$	$g_1=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}, g_2=-E_{1,1}-iE_{2,2}-iE_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-E_{2,2}+E_{3,3}+iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=189$	$g_1=-E_{1,4}-E_{2,2}+E_{3,3}-E_{4,1}, g_2=-E_{1,1}-iE_{2,3}+iE_{3,2}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-iE_{1,4}-E_{2,2}+E_{3,3}+iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=190$	$g_1=-iE_{1,4}-E_{2,3}+E_{3,2}-iE_{4,1}, g_2=-E_{1,4}-iE_{2,2}+iE_{3,3}+E_{4,1}$, $g_3=-E_{1,1}-E_{2,3}-E_{3,2}+E_{4,4}, g_4=-E_{1,1}+E_{2,3}+E_{3,2}+E_{4,4}$	$\langle c_3,c_7\rangle$	32
$j=191$	$g_1=-E_{1,4}-E_{2,3}+E_{3,2}+E_{4,1}, g_2=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}, g_4=-iE_{1,4}-E_{2,2}-E_{3,3}-iE_{4,1}$	$\langle c_3,c_7\rangle$	64
$j=192$	$g_1=-E_{1,1}-iE_{2,3}-iE_{3,2}-E_{4,4}, g_2=-E_{1,1}-E_{2,3}+E_{3,2}-E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,4}-E_{2,2}-E_{3,3}+E_{4,1}$, $g_5=-iE_{1,4}-E_{2,2}-E_{3,3}-iE_{4,1}$	$\mathcal{M}$	128
$j=193$	$g_1=-E_{1,2}-E_{2,1}-E_{3,4}-E_{4,3}, g_2=-E_{1,1}+E_{2,2}-E_{3,3}+E_{4,4}$, $g_3=-E_{1,1}-E_{2,2}+E_{3,3}+E_{4,4}, g_4=-E_{1,3}-E_{2,4}-E_{3,1}-E_{4,2}$, $g_5=-iE_{1,4}-iE_{2,3}-iE_{3,2}-iE_{4,1}$	$\mathcal{M}$	64

Table C.2. Non-abelian and pairwise non-conjugate subgroups of order $3 \cdot 2^k$ in $G_{3\cdot 512}$.

$g[j]$	Generators	$h[j] \cap \mathcal{M}$	ord
$j = 1$	$\rho, h_1 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	24
$j = 2$	$\rho, h_1 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	12
$j = 3$	$\rho, h_1 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	192
$j = 4$	$\rho, h_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	48
$j = 5$	$\rho, h_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	24
$j = 6$	$\rho, h_1 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	384
$j = 7$	$\rho, h_1 = -E_{1,1} - E_{2,3} + E_{3,2} - E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	24
$j = 8$	$\rho, h_1 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	192
$j = 9$	$\rho, h_1 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	12
$j = 10$	$\rho, h_1 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}, h_2 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle I_4 \rangle$	6
$j = 11$	$\rho, h_1 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	192
$j = 12$	$\rho, h_1 = -E_{1,1} - iE_{2,3} - iE_{3,2} - E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	96
$j = 13$	$\rho, h_1 = -E_{1,1} - iE_{2,3} + iE_{3,2} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	96
$j = 14$	$\rho, h_1 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	96
$j = 15$	$\rho, h_1 = -E_{1,1} - iE_{2,2} - iE_{3,3} + E_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\langle c_1, c_2 \rangle$	48
$j = 16$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	24
$j = 17$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	96
$j = 18$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,1} - E_{2,2} - iE_{3,3} + iE_{4,4}$	$\mathcal{M}$	768
$j = 19$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\mathcal{M}$	192
$j = 20$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,1} - E_{2,3} - iE_{3,2} - iE_{4,4}$	$\mathcal{M}$	1536
$j = 21$	$\rho, h_1 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}, h_2 = -E_{1,2} - E_{2,1} - E_{3,4} - E_{4,3}$	$\langle I_4 \rangle$	12
$j = 22$	$\rho, h_1 = -E_{1,2} - E_{2,4} - E_{3,1} + E_{4,3}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	48
$j = 23$	$\rho, h_1 = -E_{1,2} + E_{2,1} - E_{3,4} + E_{4,3}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	24
$j = 24$	$\rho, h_1 = -E_{1,2} - iE_{2,1} - E_{3,4} + iE_{4,3}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	192
$j = 25$	$\rho, h_1 = -E_{1,4} - iE_{2,3} - iE_{3,2} + E_{4,1}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	48
$j = 26$	$\rho, h_1 = -E_{1,4} - iE_{2,3} - iE_{3,2} + E_{4,1}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	192
$j = 27$	$\rho, h_1 = -E_{1,4} - iE_{2,3} - iE_{3,2} + E_{4,1}, h_2 = -E_{1,1} - E_{2,3} - E_{3,2} + E_{4,4}$	$\mathcal{M}$	384
$j = 28$	$\rho, h_1 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	48
$j = 29$	$\rho, h_1 = -E_{1,4} - iE_{2,3} + iE_{3,2} - E_{4,1}, h_2 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	96
$j = 30$	$\rho, h_1 = -E_{1,4} - iE_{2,2} - iE_{3,3} - E_{4,1}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	192
$j = 31$	$\rho, h_1 = -E_{1,4} - iE_{2,2} + iE_{3,3} + E_{4,1}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	48
$j = 32$	$\rho, h_1 = +E_{1,1} - E_{2,3} - E_{3,2} - E_{4,4}, h_2 = +E_{1,1} - E_{2,3} - E_{3,2} - E_{4,4}$	$\langle I_4 \rangle$	6
$j = 33$	$\rho, h_1 = -iE_{1,1} - iE_{2,3} - iE_{3,2} + iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\mathcal{M}$	48
$j = 34$	$\rho, h_1 = -iE_{1,1} - iE_{2,3} - iE_{3,2} + iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} + E_{4,4}$	$\mathcal{M}$	96
$j = 35$	$\rho, h_1 = -iE_{1,1} + iE_{2,3} + iE_{3,2} + iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} - E_{3,3} - E_{4,4}$	$\langle c_7 \rangle$	12
$j = 36$	$\rho, h_1 = -iE_{1,1} + iE_{2,3} + iE_{3,2} + iE_{4,4}, h_2 = -E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4}$	$\langle c_7 \rangle$	24
$j = 37$	$\rho, h_1 = -iE_{1,1} - iE_{2,2} + iE_{3,3} - iE_{4,4}, h_2 = -E_{1,1} - E_{2,2} + E_{3,3} + E_{4,4}$	$\mathcal{M}$	48

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