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GROUPS OF ORDER 32 AS GALOIS GROUPS*

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Abstract. We find the obstructions to realizability of groups of order 32 as Galois groups over arbitrary field of characteristic not 2. We discuss explicit extensions and automatic realizations as well.

1. Introduction. Let k be arbitrary field of characteristic not 2. In this article we discuss certain embedding problems with kernel of order 2 or 4. First, let us recall the general description of the embedding problem. Let G be a finite group and let

$$1 \to A \to G \underset{\psi}{\to} F \to 1$$

be a finite group extension. Let also K/k be a Galois extension with Galois group F. The embedding problem then consists in determining whether there exists a Galois extension L such that $K \subset L, G \cong \mathrm{Gal}(L/k)$ and for all $g \in G$ the restriction $g|_K$ equals $\psi(g)$. The embedding problem we denote by $(K/k, G, A)$.

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The group A we call *the kernel* of the embedding problem. Now, let us restrict ourselves to the embedding problem with kernel $\mu_2 = \{\pm 1\}$ of order 2, so we can regard μ_2 as the group of the square roots of unity, contained in the multiplicative group $K^\times$. Then the injection $\mu_2 \to K^\times$ induces the map:

$$i : H^2(F, \mu_2) \to H^2(F, K^\times).$$

Denote by γ the 2-coclass of the group extension

$$1 \to \mu_2 \to G \underset{\psi}{\to} F \to 1 \tag{1.1}$$

in $H^2(F, \mu_2)$.

The element $i(\gamma)$ is called *the obstruction* to solvability of the embedding problem (or simply to realizability of the group G as a Galois group over k). Given that the extension (1.1) is nonsplit, the embedding problem $(K/k, G, \mu_2)$ is solvable if and only if $i(\gamma) = 1$ (the cohomological groups are written multiplicatively), see [5], for example.

Let $c \in Z^2(F, \mu_2)$ represent γ. It is well known that $H^2(F, K^\times)$ is isomorphic to the relative Brauer group $\mathrm{Br}(K/k)$ of K/k by $i(\gamma) \mapsto [K, F, c]$, where $[K, F, c] \in \mathrm{Br}(K/k)$ is the equivalence class of the crossed product algebra (K, F, c), i.e., (K, F, c) is a central simple algebra over k, generated by K and elements u_σ with relations $u_1 = c_{1,1}, u_\sigma u_\tau = c_{\sigma,\tau} u_{\sigma\tau}$ and $u_\sigma x = \sigma(x) u_\sigma$, for $\sigma, \tau \in F$ and $x \in K$.

It is well known also that the absolute Brauer group $\mathrm{Br}(k)$ is identified with $\varinjlim \mathrm{Br}(K/k)$, where K/k runs through all finite Galois extensions. Since γ is an element of order 2, the obstruction $i(\gamma)$ lies in the 2-torsion of the Brauer group $\mathrm{Br}(k)$. By Merkurjev's Theorem [10] the obstruction can be written as a product of classes of quaternion algebras. One of our goals is to find these products for each group under consideration.

Another goal is to describe all Galois extensions, solving the embedding problem $(K/k, G, \mu_2)$. That can be achieved in the following manner. Assume that the obstruction is split, i.e., $i(\gamma) = 1$. Then $c \in B^2(F, K^\times)$, i.e., there exists a map $a : F \to K^\times$, such that $c_{\sigma,\tau} = a_\sigma \sigma a_\tau a_{\sigma\tau}^{-1}, \forall \sigma, \tau \in F$. Since $c_{\sigma,\tau}$ is in μ_2, we have that $\sigma \mapsto a_\sigma^2$ is a crossed homomorphism $F \to K^\times$. Then by Hilbert's Theorem 90, there exists an $\omega \in K^\times$ such that $\sigma\omega/\omega = a_\sigma^2, \forall \sigma \in F$. This is part of the proof of the following theorem, proven in [5] not only for group extensions with kernel μ_2, but also for central group extensions with kernel μ_p, for an odd prime p.

Theorem 1.1. *In the above notations, all solutions to the embedding problem $(K/k, G, \mu_2)$ are $K(\sqrt{r\omega})/k$, where r runs through $k^\times$.*

In the light of these observations we may proceed by the following scheme, when looking for the element ω:

(1) Check whether $\sigma\omega/\omega$ is in $K^{\times 2}, \forall \sigma \in F$. This will guarantee that $K(\sqrt{r\omega})/k$ is Galois. To this end it is enough to consider only a minimal generating set of the group F.

(2) Take arbitrary preimages of the generating set in the group $\mathrm{Gal}(K(\sqrt{r\omega})/k)$. Check that they fulfil the relations defining the group G. That this is enough is explained in the introduction of the paper [9]. For example, if $\sigma \in F$ is of order k, the preimage of σ, say $\tilde{\sigma} \in \mathrm{Gal}(K(\sqrt{r\omega})/k)$, is of order at most $2k$. We always may put $\tilde{\sigma}\sqrt{r\omega} = \sqrt{r\omega}a_\sigma$, whence $\tilde{\sigma}^k\sqrt{r\omega} = \sqrt{r\omega}a_\sigma \sigma a_\sigma \cdots \sigma^{k-1}a_\sigma$. Therefore, $\tilde{\sigma}$ is of order k iff $a_\sigma \sigma a_\sigma \cdots \sigma^{k-1}a_\sigma = 1$ and of order $2k$ iff $a_\sigma \sigma a_\sigma \cdots \sigma^{k-1}a_\sigma = -1$.

In works such as [7], [2] the obstructions to realizability of the groups of orders 8 and 16 are expressed as products of quaternion classes. In [8], [11], [12] and [14] are considered several groups of order 32. Some of the obstructions to realizability of these groups are given at the end of our work. We will not consider the cyclic group C_{32}, for which our methods are inapplicable. We will not consider also groups which are a direct product of groups : $G \times H$, since their realizability depends solely on the realizability of the groups G and H. In this way, it remains to calculate the obstructions of 27 groups out of the total number of 51 groups of order 32. We employ the computer program GAP 3 to list the presentations of the groups and some other details in the appendix. Minimal presentations for the groups of order 32 can be found also in [13] and [3].

2. Groups of orders 8 and 16. We will need some notations about the groups of orders 8 and 16 and also several criteria, which are found in [7]. The notations in this section are used throughout all the work, unless otherwise stated.

The dihedral group D_8 of order 8 is generated by elements σ and τ such that $\sigma^4 = \tau^2 = 1$ and $\tau\sigma = \sigma^3\tau$. The full set of D_8 extensions is described thus: Let a and b be quadratically independent over k such that $(a, ab) = 1 \in \mathrm{Br}(k)$, and let $\alpha_1 \in k$ and $\alpha_2 \in k^\times$ be such that $ab = \alpha_1^2 - a\alpha_2^2$. Then

$$K/k = k\left(\sqrt{r(\alpha_1 - \alpha_2\sqrt{a})}, \sqrt{b}\right)/k$$

is a D_8 Galois extension, for all $r \in k^\times$. Put $\alpha = \alpha_1 - \alpha_2\sqrt{a}$ and $\alpha' = \alpha_1 + \alpha_2\sqrt{a}$. Then we can assume that σ and τ act in this way:

$$\begin{aligned} \sigma &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha'},\ \sqrt{r\alpha'} \mapsto -\sqrt{r\alpha},\ \sqrt{b} \mapsto \sqrt{b}; \\ \tau &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha},\ \sqrt{r\alpha'} \mapsto -\sqrt{r\alpha'},\ \sqrt{b} \mapsto -\sqrt{b}. \end{aligned}$$

The quaternion group Q_8 of order 8 is generated by elements σ and τ, such that $\sigma^4 = \tau^4 = 1, \sigma^2 = \tau^2$ and $\tau\sigma = \sigma^3\tau$. We will not need the description of Q_8 extensions.

The group $C_4 \times C_2$ is generated by two elements, say, ρ_1 and ρ_2, such that $\rho_1^4 = \rho_2^2 = 1$ and $\rho_1\rho_2 = \rho_2\rho_1$. The full set of $C_4 \times C_2$ extensions is described thus: Let a and b be quadratically independent over k such that $(a,a) = 1 \in \mathrm{Br}(k)$ and let $c \in k^\times$ be such that $a = 1 + c^2$. Then

$$K/k = k\left(\sqrt{r(a+\sqrt{a})}, \sqrt{b}\right)/k$$

is a $C_4 \times C_2$ extension, for all $r \in k^\times$. Assume also that ρ_1 and ρ_2 act in this way:

$$\begin{aligned} \rho_1 &: \sqrt{r(a+\sqrt{a})} \mapsto \sqrt{r(a-\sqrt{a})},\ \sqrt{b} \mapsto \sqrt{b}; \\ \rho_2 &: \sqrt{r(a+\sqrt{a})} \mapsto \sqrt{r(a+\sqrt{a})},\ \sqrt{b} \mapsto -\sqrt{b}. \end{aligned}$$

The non abelian groups of order 16 are: M_{16}-*the modular* group, SD_{16}-*the semidihedral (quasidihedral)* group (also denoted as SD_8 and QD_8), D_{16}-*the dihedral* group (also denoted as D_8), Q_{16}-*the quaternion* group, $Q \curlywedge C$-*the pullback* of the homomorphisms $Q_8 \mapsto C_2$ and $C_4 \mapsto C_2$, $D \curlywedge C$- the pullback of the homomorphisms $D_8 \mapsto C_2$ and $C_4 \mapsto C_2$, DC-*the central product* of D_8 and C_4, $D_8 \times C_2$ and $Q_8 \times C_2$. Their presentations by a set of generators are:

$$\begin{aligned} &M_{16} \cong \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \tau\sigma = \sigma^5\tau \rangle, \\ &SD_{16} \cong \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \tau\sigma = \sigma^3\tau \rangle, \\ &D_{16} \cong \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle, \\ &Q_{16} \cong \langle \sigma, \tau \mid \sigma^8 = 1, \tau^2 = \sigma^4, \tau\sigma = \sigma^{-1}\tau \rangle, \\ &Q \curlywedge C \cong \langle \sigma, \tau \mid \sigma^4 = \tau^4 = 1, \tau\sigma = \sigma^3\tau \rangle, \\ &D \curlywedge C \cong \langle \sigma, \tau, \rho \mid \sigma^4 = \tau^2 = \rho^2 = 1, \tau\sigma = \sigma^3\tau\rho, [\sigma,\rho] = [\tau,\rho] = 1 \rangle, \\ &DC \cong \langle \sigma, \tau, \rho \mid \sigma^4 = \tau^2 = 1, \tau\sigma = \sigma^3\tau, \sigma^2 = \rho^2, [\sigma,\rho] = [\tau,\rho] = 1 \rangle. \end{aligned}$$

We will use the following criteria, proven in [7].

Theorem 2.1. *Let $K/k = k(\sqrt{a_1}, \dots, \sqrt{a_n})/k$ be a C_2^n extension, and let $\sigma_1, \dots, \sigma_n \in C_2^n$ be given by $\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$. Let*

$$1 \to \mu_2 \to G \to C_2^n \to 1 \tag{2.1}$$

be a non split extension, and choose pre-images $s_1, \dots, s_n \in G$ to $\sigma_1, \dots, \sigma_n$. Define d_{ij}, $i \le j$, by $s_i^2 = (-1)^{d_{ii}}$ and $s_i s_j = (-1)^{d_{ij}} s_j s_i$, $i < j$. Then the obstruction to the embedding problem given by K/k and (2.1) is

$$\prod_{i \le j} (a_i, a_j)^{d_{ij}} \in \mathrm{Br}(k).$$

(Here we use the standard notation (a_i, a_j) of the quaternion class in $\mathrm{Br}(k)$*.)*

Theorem 2.2. *Let K/k be an $C_4^r \times C_2^s$ extension. We can write:*

$$K = k(\sqrt{q_1(a_1 + \sqrt{a_1})}, \dots, \sqrt{q_r(a_r + \sqrt{a_r})}, \sqrt{a_{r+1}}, \dots, \sqrt{a_{r+s}}),$$

where $a_1, \dots, a_{r+s} \in k^\times$ are quadratically independent, $a_i = 1 + c_i^2$ for $i \le r$, and $q_i \in k^\times$. Let $\rho_1, \dots, \rho_{r+s} \in \mathrm{Gal}(K/k)$, such that $\rho_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$. Let

$$1 \to \mu_2 \to G \to C_4^r \times C_2^s \to 1 \tag{2.2}$$

be a non split extension, and choose pre-images $t_1, \dots, t_{r+s} \in G$ to $\rho_1, \dots, \rho_{r+s}$. Then the obstruction to the embedding problem given by K/k and (2.2) is:

$$\prod_{i=r+1}^{r+s} (a_i, a_i)^{d_i} \cdot \prod_{i=1}^{r} [(a_i, 2)(-1, q_i)]^{d_i} \cdot \prod_{i<j} (a_i, a_j)^{d_{ij}},$$

where $t_i^2 = (-1)^{d_i}$ for $i > r$, $t_i^4 = (-1)^{d_i}$ for $i \le r$, and $t_i t_j = (-1)^{d_{ij}} t_j t_i$.

Theorem 2.3. *Let K/k be a D_8 extension as described above, and let*

$$1 \to \mu_2 \to G \to D_8 \to 1 \tag{2.3}$$

be a non split extension, and choose pre-images s and t in G of σ and τ respectively. Then the obstruction to the embedding problem given by K/k and (2.3) is:

$$[(a, -2)(-b, 2\alpha_1 r)]^i (b, -1)^j (a, -1)^k \in \mathrm{Br}(k),$$

where $s^4 = (-1)^i, t^2 = (-1)^j$ *and* $ts = (-1)^k s^3 t$.

Now, we extend the latter criterion for the group $D_8 \times C_2$, generated by σ, τ and ρ, such that $\sigma^4 = \tau^2 = \rho^2 = 1, \tau\sigma = \sigma^3\tau$ and ρ is central. We may also apply [12], Theorem 4.1, to obtain:

Theorem 2.4. *Let* $K/k = k(\sqrt{r\alpha}, \sqrt{b}, \sqrt{c})/k$ *be a* $D_8 \times C_2$ *extension and let*

$$1 \to \mu_2 \to G \to D_8 \times C_2 \to 1 \tag{2.4}$$

be a non split extension, and choose pre-images s, t *and* p *of* σ, τ *and* ρ *respectively. Then the obstruction to the embedding problem given by* K/k *and* (2.4) *is:*

$$[(a, -2)(-b, 2\alpha_1 r)]^i (b, -1)^j (a, -1)^k (c^l a^{d_1} b^{d_2}, c),$$

where $s^4 = (-1)^i$, $t^2 = (-1)^j$, $ts = (-1)^k s^3 t$, $p^2 = (-1)^l$, $ps = (-1)^{d_1} sp$, $pt = (-1)^{d_2} tp$.

3. The groups of order 32. We write in a table in the appendix the relations between the generators of all groups of order 32, the rank (i.e. the minimal number of generators of the quotient group by the Frattini subgroup), the centre and the exponent. In the notations of GAP 3, each group G_i ($i = 1, \dots, 51$) is generated by 5 elements: $a_1, \dots, a_5$. We put $[a_i, a_j] = a_i^{-1} a_j^{-1} a_i a_j$ – the commutator of the two elements a_i and a_j. The appearance of certain expression in the field with relations means that it is equal to 1. In order to write less, we skip the commutators in which one of the elements lies in the centre. For example, the element a_5 is in the centre for each group G_i, so we need not write the commutators of the kind $[a_i, a_5]$.

The 24 groups, for which we will not calculate the obstructions are: the abelian groups – $G_1, G_3, G_{16}, G_{21}, G_{36}, G_{45}, G_{51}$; the non abelian groups of exponent 16 – G_{17}, G_{18}, G_{19} and G_{20}; the non abelian groups of the kind $H \times C_2$ – $G_{22}, G_{23}, G_{37}, G_{39}, G_{40}, G_{41}, G_{46}, G_{47}$ and G_{48}; the non abelian groups of the kind $H \times C_4$ – G_{24} and G_{25}; and the extra-special groups G_{49} and G_{50}.

4. The pullbacks. Let $\varphi' : G' \to F$ and $\varphi'' : G'' \to F$ be homomorphisms with kernels N' and, respectively, N''. *The pullback* of the pair of homomorphisms φ' and φ'' is called the subgroup in $G' \times G''$ of all pairs (σ', σ''), such that $\varphi'(\sigma') = \varphi''(\sigma'')$. The pullback is denoted by $G' \curlywedge G''$. It is also called the direct product of the groups G' and G'' with amalgamated quotient group F and denoted by $G' *_F G''$.

Now, let $N_1 = N' \times \{1\}$ and $N_2 = \{1\} \times N''$. Then N_1 and N_2 are normal subgroups of $G' \curlywedge G''$, such that $N_1 \cap N_2 = \{1\}$. The converse is also true (see [4], I, §12):

Lemma 4.1. *Let N_1 and N_2 be two normal subgroups of the group G, such that $N_1 \cap N_2 = \{1\}$. Then G is isomorphic to the pullback $(G/N_1) \curlywedge (G/N_2)$. Also, we have the commutative diagram:*

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N_2 & = & N_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N_1 & \longrightarrow & G & \xrightarrow{\varphi_1} & G/N_1 & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2^*} & & \\
1 & \longrightarrow & N_1 & \longrightarrow & G/N_2 & \xrightarrow{\varphi_1^*} & G/N_1N_2 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

where a homomorphism of a group onto a quotient group is natural.

The application to embedding problems is given by:

Theorem 4.1. *Let K/k be a Galois extension with Galois group F. In the notations of the lemma, let $F \cong G/N_1N_2$ and $G \cong (G/N_1) \curlywedge (G/N_2)$. Then the embedding problem $(K/k, G, N_1 \times N_2)$ is solvable iff the embedding problems $(K/k, G/N_1, N_2)$ and $(K/k, G/N_2, N_1)$ are solvable.*

Since we will consider groups of order 32, we will be looking for normal subgroups N_1 and N_2 of order 2. In that case, the group G is a pullback iff the centre $Z(G)$ has at least two elements of order 2 (in other words, $Z(G)$ is not cyclic). The pullbacks, which we will discuss are 18: G_2, G_4, G_5, G_9, G_{10}, G_{12}, G_{13}, G_{14}, $G_{26} - G_{35}$.

Let us begin with the group G_2. We give all the details for this group as an example.

4.1. The group G_2. The centre $Z(G_2) = \langle a_3, a_4, a_5 \rangle$ is isomorphic to C_2^3. Let $N_1 = \langle a_4 \rangle$, $N_2 = \langle a_5 \rangle$ and $N = N_1N_2 = N_1 \times N_2$. Then the quotient group G_2/N is isomorphic to D_8. Consider the embedding problem given by K/k with Galois group D_8 (according to the notations in Section 2) and the group extension

$$1 \to N \to G_2 \underset{\substack{a_1 \mapsto \sigma_1 \\ a_2 \mapsto \tau_1}}{\longrightarrow} D_8 \to 1,$$

where $\sigma_1^2 = \tau_1^2 = [\sigma_1, \tau_1]^2 = 1$. Let $\sigma = \sigma_1\tau_1$ and $\tau = \tau_1$. Then $|\sigma| = 4$, $|\tau| = 2$ and $\tau\sigma = \sigma^3\tau$.

Now, consider the embedding problem given by K/k and the group extension

$$1 \to N_2 \to G_2/N_1 \underset{\substack{b_1 b_2 \mapsto \sigma \\ b_2 \mapsto \tau}}{\longrightarrow} D_8 \to 1.$$

The group G_2/N_1 is generated by elements $b_i = a_i\langle a_4\rangle \in G_2/N_1, i \neq 4$ such that $b_1^2 = b_3^2 = b_5^2 = 1$, $b_2^2 = b_5$, $[b_2, b_1] = b_3$, b_3 is central, whence G_2/N_1 is isomorphic to $D \curlywedge C$. Also, we have the relations $(b_1b_2)^2 = b_5b_3$, $(b_1b_2)^4 = 1$ and $b_2(b_1b_2) = (b_1b_2)^3b_2b_5 = -(b_1b_2)^3b_2$. Then Theorem 2.3 implies that the obstruction to the latter embedding problem is $(ab, -1) \in \mathrm{Br}(k)$.

Now, consider G_2/N_2, which is generated by elements $b_i = a_i\langle a_5\rangle \in G_2/N_2$, $i = 1, \dots, 4$, such that $b_1^4 = b_2^2 = b_3^2 = b_4^2 = 1$, $b_4 = b_1^2$, $[b_1, b_2] = b_3, b_3$ is central, whence G_2/N_2 is isomorphic to $D \curlywedge C$. We have the relation $b_2(b_1b_2) = -(b_1b_2)^3b_2$. The obstruction then to the embedding problem given by K/k and the group extension

$$1 \to N_1 \to G_2/N_2 \underset{\substack{b_1 b_2 \mapsto \sigma \\ b_2 \mapsto \tau}}{\longrightarrow} D_8 \to 1.$$

is $(a, -1) \in \mathrm{Br}(k)$.

Thus, we obtain that the embedding problem $(K/k, G_2, N)$ is solvable iff $(ab, -1) = (a, -1) = 1 \in \mathrm{Br}(k)$, where $a, b \in k^\times$ are quadratically independent such that $(a, ab) = 1 \in \mathrm{Br}(k)$ (a necessary condition).

The remaining groups can be investigated in the same way. We write down only the main points in our calculations.

4.2. The group G_4. $Z(G_4) = \langle a_3, a_4, a_5\rangle = \langle a_3, a_4\rangle \cong C_4 \times C_2$, $N_1 = \langle a_4\rangle$, $N_2 = \langle a_5\rangle$, $N = N_1 \times N_2$, $G_4/N_1 \cong M_{16}, G_4/N_2 \cong C_4 \times C_4$. The embedding problem $(K/k, G_4, N)$ given by a $C_4 \times C_2$ extension K/k and the group extension

$$1 \to N \to G_4 \underset{\substack{a_1 \mapsto \rho_1 \\ a_2 \mapsto \rho_2}}{\longrightarrow} C_4 \times C_2 \to 1.$$

is solvable iff $(a,2b)(-1,r)=(b,b)=1\in \mathrm{Br}(k)$, where $(a,a)=1$ is necessary for the existence of a C_4 extension.

4.3. The group G_5. $Z(G_5)=\langle a_3,a_4,a_5\rangle=\langle a_3,a_4\rangle\cong C_2\times C_4$, $N_1=\langle a_3\rangle$, $N_2=\langle a_5\rangle$, $N=N_1\times N_2$, $G_5/N_1\cong C_8\times C_2$, $G_5/N_2\cong D\curlywedge C$. The embedding problem $(K/k,G_5,N)$ given by a $C_4\times C_2$ extension K/k and the group extension

$$1\to N\to G_5 \underset{\substack{a_1\mapsto\rho_1\\ a_2\mapsto\rho_2}}{\longrightarrow} C_4\times C_2\to 1.$$

is solvable iff $(a,2)(-1,r)=(a,b)=1\in \mathrm{Br}(k)$, where $(a,a)=1$ is a necessary condition.

4.4. The group G_9. $Z(G_9)=\langle a_4,a_5\rangle\cong C_2^2$, $N_1=\langle a_4\rangle$, $N_2=\langle a_5\rangle$, $N=N_1\times N_2$, $G_9/N_1\cong D_{16}$, $G_9/N_2\cong D\curlywedge C$. The embedding problem $(K/k,G_9,N)$ given by a D_8 extension K/k and the group extension

$$1\to N\to G_9 \underset{\substack{a_1a_2\mapsto\sigma\\ a_2\mapsto\tau}}{\longrightarrow} D_8\to 1.$$

is solvable iff $(ab,2)(-b,\alpha_1 r)=(a,a)=1\in \mathrm{Br}(k)$, where $(a,ab)=1$ is a necessary condition.

4.5. The group G_{10}. $Z(G_{10})=\langle a_4,a_5\rangle\cong C_2^2$, $N_1=\langle a_4\rangle$, $N_2=\langle a_5\rangle$, $N=N_1\times N_2$, $G_{10}/N_1\cong SD_{16}$, $G_{10}/N_2\cong D\curlywedge C$. The embedding problem $(K/k,G_{10},N)$ given by a D_8 extension K/k and the group extension

$$1\to N\to G_{10} \underset{\substack{a_1a_2\mapsto\sigma\\ a_2\mapsto\tau}}{\longrightarrow} D_8\to 1.$$

is solvable iff $(a,-2)(-b,2\alpha_1 r)=(a,a)=1\in \mathrm{Br}(k)$, where $(a,ab)=1$ is a necessary condition.

4.6. The group G_{12}. $Z(G_{12})=\langle a_3,a_4,a_5\rangle=\langle a_3,a_4\rangle\cong C_2\times C_4$, $N_1=\langle a_3\rangle$, $N_2=\langle a_5\rangle$, $N=N_1\times N_2$, $G_{12}/N_1\cong C_8\times C_2$, $G_{12}/N_2\cong Q\curlywedge C$. The embedding problem $(K/k,G_{12},N)$ given by a $C_4\times C_2$ extension K/k and the group extension

$$1\to N\to G_{12} \underset{\substack{a_1\mapsto\rho_1\\ a_2\mapsto\rho_2}}{\longrightarrow} C_4\times C_2\to 1.$$

is solvable iff $(a,2)(-1,r)=(ab,b)=1\in \mathrm{Br}(k)$, where $(a,a)=1$ is a necessary condition.

4.7. The group G_{13}. $Z(G_{13}) = \langle a_4, a_5\rangle \cong C_2^2, N_1 = \langle a_4\rangle$, $N_2 = \langle a_5\rangle$, $N = N_1 \times N_2$, $G_{13}/N_1 \cong SD_{16}$, $G_{13}/N_2 \cong Q \curlywedge C$. The embedding problem $(K/k, G_{13}, N)$ given by a D_8 extension K/k and the group extension

$$1 \to N \to G_{13} \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma}}{\longrightarrow} D_8 \to 1.$$

is solvable iff $(a, -2)(-b, 2\alpha_1 r) = (b, b) = 1 \in \mathrm{Br}(k)$, where $(a, ab) = 1$ is a necessary condition.

4.8. The group G_{14}. $Z(G_{14}) = \langle a_4, a_5\rangle \cong C_2^2, N_1 = \langle a_4\rangle$, $N_2 = \langle a_5\rangle$, $N = N_1 \times N_2$, $G_{14}/N_1 \cong D_{16}$, $G_{14}/N_2 \cong Q \curlywedge C$. The embedding problem $(K/k, G_{14}, N)$ given by a D_8 extension K/k and the group extension

$$1 \to N \to G_{14} \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma}}{\longrightarrow} D_8 \to 1.$$

is solvable iff $(ab, 2)(-b, \alpha_1 r) = (b, b) = 1 \in \mathrm{Br}(k)$, where $(a, ab) = 1$ is a necessary condition.

For each of the remaining groups $G_{26} - G_{35}$ we put $N_1 = \langle a_4\rangle$, $N_2 = \langle a_5\rangle$ and $N = N_1 \times N_2$. The quotient group G_i/N is isomorphic to C_2^3. Therefore we consider the embedding problem given by a C_2^3 extension $K/k = k(\sqrt{a}, \sqrt{b}, \sqrt{c})/k$ and the group extension

$$1 \to N \to G_i \underset{\substack{a_1 \mapsto \sigma_1 \\ a_2 \mapsto \sigma_2 \\ a_3 \mapsto \sigma_3}}{\longrightarrow} C_2^3 \to 1,$$

for $i = 26, \ldots 35$. Now, we can apply Theorem 2.1. The obstructions to solvability of the embedding problems $(K/k, G_i, N)$ are given in Table 1.

5. Groups having a quotient group ot the kind $H \times C_2$.

There are four groups, having a quotient group ot the kind $H \times C_2$: G_{38}, G_{42}, G_{43} and G_{44}.

5.1. The group G_{38}. The centre $Z(G_{38}) = \langle a_1, a_4, a_5\rangle = \langle a_1\rangle$ is isomorphic to the cyclic group C_8 and the quotient group $G_{38}/\langle a_5\rangle$ is isomorphic to $C_4 \times C_2^2$. Let a, b and c be quadratically independent and $(a, a) = 1 \in \mathrm{Br}(k)$. Then $K/k = k(\sqrt{r(a + \sqrt{a})}, \sqrt{b}, \sqrt{c})/k$ is a $C_4 \times C_2^2$ extension for all $r \in k^\times$.

Table 1

i	obstructions
26	$(ac, ac), (ab, b)(c, c)$
27	$(a, c), (a, b)$
28	$(a, c), (b, ab)$
29	$(a, c), (a, ab)(b, b)$
30	$(a, c), (c, c)(a, b)$
31	$(b, b)(a, c), (c, c)(a, b)$
32	$(b, b)(a, c), (a, a)(c, c)(a, b)$
33	$(b, b)(a, c), (b, b)(c, c)(a, b)$
34	$(ac, c), (ab, b)$
35	$(ac, c), (a, ab)(b, b)$

From Theorem 2.2 follows that the embedding problem given by K/k and the group extension

$$1 \to \mu_2 \cong \langle a_5 \rangle \to G_{38} \underset{\substack{a_1 \mapsto \rho_1 \\ a_2 \mapsto \rho_2 \\ a_3 \mapsto \rho_3}}{\longrightarrow} C_4 \times C_2^2 \to 1$$

is solvable iff

$$(a, 2)(-1, r)(b, c) = 1 \in \mathrm{Br}(k).$$

For the remaining three groups we have that the quotient group by the cyclic subgroup $\mu_2 \cong \langle a_5 \rangle$ is isomorphic to the group $D_8 \times C_2 \cong \langle \sigma, \tau \rangle \times \langle \rho \rangle$, so we apply Theorem 2.4. Now, we discuss the following three embedding problems given by a $D_8 \times C_2$ extension K/k and the group extensions

$$1 \to \mu_2 \cong \langle a_5 \rangle \to G_i \underset{\substack{a_2 a_1 \mapsto \sigma \\ a_2 \mapsto \tau \\ a_3 \mapsto \rho}}{\longrightarrow} D_8 \times C_2 \to 1,$$

for $i = 42, 43, 44$. In all three cases we have that $(a, ab) = 1$ is a necessary condition in order to construct the embedding problems. The obstructions to solvability of the embedding problems $(K/k, G_i, \langle a_5 \rangle)$ are given in Table 2.

We note that the obstruction for $i = 43$ is a product of two quaternion algebras. Therefore, at this point we can turn our attention to Galois extensions, realizing the group G_{43}. Firstly, we give a parametrization of all G_{43} extensions in the general case, when $b \neq_2 -1$, i.e. b and -1 are quadratically independent

Table 2

i	obstructions
42	$(a,2)(-b,2\alpha_1 r)(c,c)$
43	$(a,2c)(-b,2\alpha_1 r)$
44	$(a,-2c)(-b,2\alpha_1 r)(b,b)$

mod $k^{\times 2}$ (for abuse of notation we will use the symbol $=_2$ to denote that two elements are quadratically dependent, and $\neq_2$ if they are independent).

Let us give before that some notations, following [9]. For $a, b \in k^\times$, the *quaternion algebra* $(a, b/k)$ is the k-algebra generated by elements α and β with relations $\alpha^2 = a$, $\beta^2 = b$ and $\beta\alpha = -\alpha\beta$. The equivalence class as an element in the Brauer group $\mathrm{Br}(k)$ we denote by (a, b). To the quaternion algebra we associate the quadratic form in canonic type $\langle a, b, -ab\rangle = ax^2 + by^2 - abz^2$. Then $(a, b/k)$ is split iff $\langle a, b, -ab\rangle$ is isotropic (i.e., represents 0). Two quaternion algebras $(a, b/k)$ and $(c, d/k)$ are isomorphic, iff the quadratic forms $\langle a, b, -ab\rangle$ and $\langle c, d, -cd\rangle$ are equivalent. For an abuse of notation we will denote by $\langle a, b, -ab\rangle$ also the diagonal matrix $\mathrm{diag}(a, b, -ab)$. Then the equivalence of the quadratic forms $\langle a, b, -ab\rangle$ and $\langle c, d, -cd\rangle$ is expressed by the matrix equation $\mathbf{P}^t\langle a, b, -ab\rangle\mathbf{P} = \langle c, d, -cd\rangle$, for some non-singular 3×3 matrix $\mathbf{P}$ over k.

Theorem 5.1. *Let K/k be a $D_8 \times C_2$ extension as above, and assume $\alpha_1 \neq 0$. Then the embedding problem $(K/k, G_{43}, \langle a_5\rangle)$ is solvable iff the quadratic forms $\langle b, r\alpha_1 c, br\alpha_1 c\rangle$ and $\langle ab, 2ca, 2bc\rangle$ are equivalent over k. If this equivalence is expressed by the matrix $\mathbf{Q}$, i.e., if*

$$\mathbf{Q}^t\langle b, r\alpha_1 c, br\alpha_1 c\rangle\mathbf{Q} = \langle ab, 2ca, 2bc\rangle,$$

we may assume $\det\mathbf{Q} = 2a/\alpha_1 r$ *and get the solutions*

$$K(\sqrt{s\omega})/k = k(\sqrt{s\omega}, \sqrt{b}, \sqrt{c})/k, \ \ s \in k^\times,$$

where

$$\omega = 1 - q_{11}/\sqrt{a} + \frac{1}{2}(q_{32} + q_{23}/\sqrt{a})\sqrt{r\alpha} + \frac{1}{2}(q_{22}/b - q_{33}/\sqrt{a})\sqrt{r\alpha'}/\sqrt{a}.$$

P r o o f. The obstruction to the embedding problem is $(a, 2c)(-b, 2r\alpha_1)$, which is equivalent to $(-ab, -2ca)(-b, -r\alpha_1 c) \in \mathrm{Br}(k)$. This gives the criterion.

We now restrict ourselves to the embedding problem given by $K/k(\sqrt{c})$ and the group extension

$$1 \to \mu_2 \cong \langle a_5 \rangle \to D_{16} \cong \langle a_2a_1, a_2 \rangle \underset{\substack{a_2a_1 \mapsto \sigma \\ a_2 \mapsto \tau}}{\longrightarrow} D_8 \to 1.$$

Define the matrix $\mathbf{P}$ with entries from $k(\sqrt{c})$:

$$\mathbf{P} = \langle 1, \sqrt{c}, \sqrt{c} \rangle \mathbf{Q} \langle 1, 1/\sqrt{c}, 1/\sqrt{c} \rangle,$$

so that we get

$$\mathbf{P}^t \langle b, r\alpha_1, br\alpha_1 \rangle \mathbf{P} = \langle ab, 2a, 2b \rangle.$$

Since the subgroup generated by a_2a_1 and a_2 is isomorphic to D_{16}, we obtain the criterion given in [9]. Then $K(\sqrt{s\omega})/k(\sqrt{c})$, for $s \in k^\times$ are the solutions to the embedding problem $(K/k(\sqrt{c}), D_{16}, \mu_2)$, where

$$\omega = 1 - p_{11}/\sqrt{a} + \frac{1}{2}(p_{32} + p_{23}/\sqrt{a})\sqrt{r\alpha} + \frac{1}{2}(p_{22}/b - p_{33}/\sqrt{a})\sqrt{r\alpha'}/\sqrt{a}.$$

(The entries of $\mathbf{P}$ and $\mathbf{Q}$ are p_{ij} and q_{ij}, $i, j = 1, 2, 3$.) It is easy to show that $p_{11} = q_{11}, p_{23} = q_{23}, p_{32} = q_{32}, p_{22} = q_{22}$ and $p_{33} = q_{33}$. Furthermore, $K(\sqrt{s\omega})/k$ is Galois, since $\rho\omega = \omega$. We let to the reader to check that this is exactly a G_{43} extension. □

Now, we give the description of G_{43} extensions in the special case when b and -1 are quadratically dependent mod $k^{\times 2}$.

Theorem 5.2. *Let $K/k = k(\sqrt[4]{a}, i, \sqrt{c})/k$ be a $D_8 \times C_2$ extension. Then the embedding problem $(K/k, G_{43}, \langle a_5 \rangle)$ is solvable iff*

$$\exists p, q \in k \ : \ p^2 - aq^2 = 2c.$$

In that case, the solutions are

$$K(\sqrt{s\omega})/k, \ \ s \in k^\times,$$

where $\omega = (p + q\sqrt{a})\sqrt[4]{a}$.

Proof. The obstruction to the embedding problem is $(a, 2c) \in \mathrm{Br}(k)$, so there exist $p, q \in k$, such that $p^2 - aq^2 = 2c$. Put $\omega = (p + q\sqrt{a})\sqrt[4]{a}$. Then we have $\sigma\omega/\omega = a_\sigma^2$, where

$$a_\sigma = \frac{\sqrt{c}(1+i)}{p + q\sqrt{a}} \in K;$$

$\tau\omega/\omega = 1$ and $\rho\omega/\omega = 1$. Therefore $K(\sqrt{s\omega})/k$ is Galois. Here, it is easy to show that it is a G_{43} extension. Obviously, $a_2^2 = 1$ and $a_3^2 = 1$. Next, $a_\sigma \sigma a_\sigma \sigma^2 a_\sigma \sigma^3 a_\sigma = -1$, whence $a_2 a_1$ is of order 8. Also, $a_2 a_1 \sqrt{s\omega} = a_\sigma \sqrt{s\omega}$, $a_2\sqrt{s\omega} = \pm\sqrt{s\omega}$ and $a_3\sqrt{s\omega} = \pm\sqrt{s\omega}$. Then $[a_2, a_3]\sqrt{s\omega} = \sqrt{s\omega}$, whence $[a_2, a_3] = 1$; and $[a_1, a_3]\sqrt{s\omega} = -\sqrt{s\omega}$, whence $[a_1, a_3] \neq 1$, but $[a_1, a_3]^2 = 1$. □

We need the so-called *common slot* property (see [6], Ch. III, Exercise 12) for the proof of the following Theorem.

Lemma 5.3. *Let $a, b, c, d \in k^\times$. Then $(a, b)(c, d) = 1 \in \mathrm{Br}(k) \iff \exists x \in k^\times$, such that $(a, bx) = (c, dx) = (ac, x) = 1$.*

Theorem 5.4. *Realizability of G_{44} as a Galois group over k implies the realizability of G_{43} (i.e., there is an automatic realizability $G_{44} \Rightarrow G_{43}$).*

Proof. Depending on the behavior of the elements -1 and 2 we consider the following cases.

(1) -1 and 2 are quadratically independent over k. Given that G_{44} is realizable, we have that $|k/k^{\times 2}| \geq 8$. We put $b = -1$ and $c = 2 : (a, 1) = (a, 4)(1, 2\alpha_1 r)$ $= 1$ for all a – quadratically independent with -1 and 2. Therefore, we obtain something more in this case: if $|k/k^{\times 2}| \geq 8$, the group G_{43} is realizable.

(2) $-1 \in k^{\times 2}$. Then the obstructions to realizability of G_{43} and G_{44} are identical: $(a, 2c)(b, 2\alpha_1 r) \in \mathrm{Br}(k)$.

(3) $-1 \notin k^{\times 2}, 2 \in k^{\times 2}$ and $-2 \notin k^{\times 2}$. Then G_{43} is realizable iff

$$(a, -b) = (a, c)(-b, \alpha_1 r) = 1;$$

and G_{44} is realizable iff

$$(a, -b) = (a, -c)(-b, \alpha_1 r)(b, -1) = 1.$$

Now, let G_{44} be realizable for some a, b and c. Consider the following sub-cases.

If $a =_2 -b$ then $(a, c)(-b, \alpha_1 r) = (-b, \alpha_1 r c) = 1$ for $r = \alpha_1 c$, so G_{43} is realizable.

If $a =_2 -1$, then $(-1, -b) = 1$ and $(b, -1) = (-1, -1)$. Whence

$$(a, -c)(-b, \alpha_1 r)(b, -1) = (-1, c)(-1, -1)(-b, \alpha_1 r)(b, -1) =$$
$$= (-1, c)(-b, \alpha_1 r) = 1.$$

If $b =_2 -1$, then $(a,-c)(-1,-1) = 1$. We use the common slot property: $(a,-c)(-1,-1) = 1$ iff $\exists y \in k^\times$ such that $(a,-cy) = (-1,-y) = (-a,y) = 1$. If $(a,a) = 1$, then we can put $b' = -a$ (quadratically independent with a): $(a,-b') = (a,a) = 1$ and $(a,c)(-b',\alpha_1 r) = (a,\alpha_1 rc) = 1$ for $r = \alpha_1 c$. Now, we have several possibilities:

If $-cy =_2 a$, then $(a,a) = 1$ and G_{43} is realizable as we have just shown.

If $-cy =_2 -1$, then again $(a,a) = 1$.

If $-cy \in k^{\times 2}$, then $y =_2 -c$, $(-1,-y) = (-1,c) = 1$ and $(-a,y) = (-a,-c) = 1$. In this case we can put $a' = -a$ and $c' = -c$. Then a', -1 and c' are again quadratically independent. Thus, $(a',1) = (a',c')(1,\alpha_1 r) = 1$.

If $-cy =_2 -a$, then $y =_2 ca$, $(-1,-y) = (-1,-ca) = 1$ and $(-a,ca) = (-a,c) = 1$. We can put here $a' = -a$ and obtain $(a',1) = (a',c)(1,\alpha_1 r) = 1$.

If -1, a and $-cy$ are quadratically independent, then we can put $c' = -cy : (a,c') = 1$, whence G_{43} is again realizable.

If a, b and -1 are quadratically independent, then we can put $c = -b$: $(a,-b) = (a,-b)(-b,\alpha_1 r) = 1$ for $r = \alpha_1$. The last case is:

(4) $-1 \notin k^{\times 2}$, $2 \notin k^{\times 2}$ and $-2 \in k^{\times 2}$. Then G_{43} is realizable iff

$$(a,-b) = (a,-c)(-b,-\alpha_1 r) = 1;$$

and G_{44} is realizable iff

$$(a,-b) = (a,c)(-b,-\alpha_1 r)(b,-1) = 1.$$

Now, let G_{44} be realizable for some a, b and c. Consider the following sub-cases.

If $a =_2 -b$ then $(a,-c)(-b,-\alpha_1 r) = (-b,\alpha_1 rc) = 1$ for $r = \alpha_1 c$, so G_{43} is realizable.

If $a =_2 -1$, then $(-1,-b) = 1$ and $(b,-1) = (-1,-1)$. Whence

$$\begin{aligned}&(a,c)(-b,-\alpha_1 r)(b,-1) = (-1,-c)(-1,-1)(-b,-\alpha_1 r)(b,-1) = \\ &= (-1,-c)(-b,-\alpha_1 r) = 1.\end{aligned}$$

If $b =_2 -1$, then $(a,c)(-1,-1) = 1$. Analogously to the same sub-case of case 3, we obtain that G_{43} is realizable.

And, finally:

If a, b and -1 are quadratically independent, we can put $c =_2 -1$: $(a,-c)(-b,-\alpha_1 r) = 1$ for $r = -\alpha_1$. $\square$

Theorem 5.5. *Let K/k be a $D_8 \times C_2$ extension as in Theorem 5.1 or 5.2, and let $(ab, ab) = 1 \in \mathrm{Br}(k)$. In that case $\exists \gamma_1, \gamma_2 \in k$, such that $\gamma_1^2 - ab\gamma_2^2 = ab$. Then the obstructions to solvability of the embedding problems $(K/k, G_{44}, \langle a_5 \rangle)$ and $(K/k, G_{43}, \langle a_5 \rangle)$ are identical and the solutions to the embedding problem $(K/k, G_{44}, \langle a_5 \rangle)$ are*

$$K(\sqrt{s(\gamma_1 + \sqrt{ab}\gamma_2)\omega})/k, \;\; s \in k^\times,$$

where ω is as in Theorem 5.1 or 5.2.

Proof. Put $\gamma = \gamma_1 + \sqrt{ab}\gamma_2$. Then $\sigma(\gamma\omega)/(\gamma\omega) = a_\sigma^2, \tau(\gamma\omega)/(\gamma\omega) = a_\tau^2$, $\rho(\gamma\omega)/(\gamma\omega) = 1$, where $a_\sigma \sigma a_\sigma \sigma^2 a_\sigma \sigma^3 a_\sigma = -1, a_\tau \tau a_\tau = -1$, whence $|a_2 a_1| = 8, |a_2| = 4$ and $|a_3| = 2$. Therefore $K(\sqrt{s\gamma\omega})/k$ is a Galois G_{44} extension. $\square$

6. The group G_6. The centre is $Z(G_6) = \langle a_5 \rangle \cong C_2$ and the quotient group is

$$G_6/\langle a_5 \rangle = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, [y, x] = z, [x, z] = [y, z] = 1 \rangle,$$

which is isomorphic to $D \curlywedge C$. In order to calculate the obstruction to realizability of the group G_6, we must describe all $D \curlywedge C$ extensions.

The notations in this section are slightly different from those in Section 2. Let a and b be quadratically independent over k. Let $(a, a) = 1 \in \mathrm{Br}(k)$ and assume $\alpha_1, \alpha_2 \in k$ are such that $\alpha_1^2 - a\alpha_2^2 = a$. Put $\alpha = \alpha_1 - \alpha_2\sqrt{a}$ and $\alpha' = \alpha_1 + \alpha_2\sqrt{a}$. Then $\alpha\alpha' = a$ and $K_1/k = k(\sqrt{r\alpha}, \sqrt{b})/k$ is a $C_4 \times C_2$ extension for all $r \in k^\times$. Conversely, all $C_4 \times C_2$ extensions are described in this way. The group $C_4 \times C_2$ is generated by elements σ and τ, such that $\sigma^4 = \tau^2 = 1$ and the actions are:

$$\begin{aligned} \sigma &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha'}, \;\sqrt{r\alpha'} \mapsto -\sqrt{r\alpha}, \;\sqrt{b} \mapsto \sqrt{b}; \\ \tau &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha}, \;\sqrt{r\alpha'} \mapsto \sqrt{r\alpha'}, \;\sqrt{b} \mapsto -\sqrt{b}. \end{aligned}$$

Then the obstruction of the embedding problem given by the extension K_1/k and the group extension

$$1 \to \langle z \rangle \to D \curlywedge C \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\longrightarrow} C_4 \times C_2 \to 1$$

is $(a,b) \in \mathrm{Br}(k)$. Now, let $(a,b)=1$, so there exist $\beta_1,\beta_2 \in k$, such that $a = \beta_1^2 - b\beta_2^2$. Put $\beta = \beta_1 - \beta_2\sqrt{b}$ and $\beta' = \beta_1 + \beta_2\sqrt{b}$. Then $\beta\beta' = a$ and $K_2/k = k(\sqrt{s\beta}, \sqrt{ab})/k$ is a D_8 extension for all $s \in k^\times$. The group D_8 is generated by elements σ_1 and τ_1, such that $\sigma_1^2 = \tau_1^2 = 1, |\sigma_1\tau_1| = 4$ and their actions are:

$$\begin{aligned}
\sigma_1 &: \sqrt{s\beta} \mapsto -\sqrt{s\beta},\ \sqrt{s\beta'} \mapsto \sqrt{s\beta'},\ \sqrt{ab} \mapsto -\sqrt{ab};\\
\tau_1 &: \sqrt{s\beta} \mapsto \sqrt{s\beta'},\ \sqrt{s\beta'} \mapsto \sqrt{s\beta},\ \sqrt{ab} \mapsto -\sqrt{ab};\\
\sigma_1\tau_1 &: \sqrt{s\beta} \mapsto \sqrt{s\beta'},\ \sqrt{s\beta'} \mapsto -\sqrt{s\beta},\ \sqrt{ab} \mapsto \sqrt{ab}.
\end{aligned}$$

The obstruction of the embedding problem given by the extension K_2/k and the group extension

$$1 \to \langle x^2 \rangle \to D \curlywedge C \underset{\substack{x\mapsto\sigma_1\\ y\mapsto\tau_1}}{\longrightarrow} D_8 \to 1$$

is $(a,a) \in \mathrm{Br}(k)$. Given that $(a,a) = (a,b) = 1$, we can make the composite $K = K_1K_2 = k(\sqrt{r\alpha}, \sqrt{b})k(\sqrt{s\beta}, \sqrt{ab})$. We will show that K/k is a $D \curlywedge C$ extension. Since the field K depends on r and s, we obtain in this way a description of all $D \curlywedge C$ extensions.

Clearly, K/k is a Galois extension. Now, let $x, y \in G = \mathrm{Gal}(K/k)$ be such that their restrictions on K_1 and K_2 are:

$$x|_{K_1} = \sigma, x|_{K_2} = \sigma_1; y|_{K_1} = \tau, y|_{K_2} = \tau_1.$$

Then the actions of x and y are:

$$\begin{aligned}
x &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha'},\ \sqrt{r\alpha'} \mapsto -\sqrt{r\alpha},\ \sqrt{b} \mapsto \sqrt{b},\\
&\quad \sqrt{s\beta} \mapsto -\sqrt{s\beta},\ \sqrt{s\beta'} \mapsto \sqrt{s\beta'},\ \sqrt{ab} \mapsto -\sqrt{ab};\\
y &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha},\ \sqrt{r\alpha'} \mapsto \sqrt{r\alpha'},\ \sqrt{b} \mapsto -\sqrt{b},\\
&\quad \sqrt{s\beta} \mapsto \sqrt{s\beta'},\ \sqrt{s\beta'} \mapsto \sqrt{s\beta},\ \sqrt{ab} \mapsto -\sqrt{ab}.
\end{aligned}$$

Thus we obtain what we looked for: $|x| = 4, |y| = 2$ and the elements of K_2 are fixed under the action of x^2. Now, put $z = [y, x]$. Then the action of z is:

$$\begin{aligned}
z &: \sqrt{r\alpha} \mapsto \sqrt{r\alpha},\ \sqrt{r\alpha'} \mapsto \sqrt{r\alpha'},\ \sqrt{b} \mapsto \sqrt{b},\\
&\quad \sqrt{s\beta} \mapsto -\sqrt{s\beta},\ \sqrt{s\beta'} \mapsto -\sqrt{s\beta'},\ \sqrt{ab} \mapsto \sqrt{ab}.
\end{aligned}$$

Therefore $|z| = 2$ and the elements of K_1 are fixed under the action of z. Also, it is easy to check that $[z,x] = [z,y] = 1$. Whence we obtain that K/k is a $D \curlywedge C$ extension and all $D \curlywedge C$ extensions are described in this way.

Now, let $E = k(\sqrt{a}, \sqrt{b})$ and let $\gamma = -(\alpha_1+\beta_1)+\alpha_2\sqrt{a}+\beta_2\sqrt{b}$. Then for the norm map N we obtain the equations $N_{E/k(\sqrt{a})}(\gamma) = d\alpha$ and $N_{E/k(\sqrt{b})}(\gamma) = d\beta$, where $d = 2(\alpha_1 + \beta_1)$.

Consider now the embedding problem given by the extension $K/k = K_1K_2/k$, described above and the group extension

$$1 \to \mu_2 \cong \langle a_5 \rangle \to G_6 \underset{\substack{a_1 \mapsto x \\ a_2 \mapsto y}}{\longrightarrow} D \curlywedge C \to 1.$$

Denote by $\Gamma = (K, D \curlywedge C, -1)$ the crossed product algebra, corresponding to the latter group extension. The dimension of Γ is $16^2 = 4^4$ and Γ can be decomposed as a tensor product of 4 quaternion algebras. The algebra Γ is generated by the following elements over k: u_x, u_y, u_z (corresponding to x, y and z); $\sqrt{r\alpha}, \sqrt{s\beta}, \sqrt{a}$ and $\sqrt{b}$. The relations in G_6 imply the following relations in Γ (recall that -1 in Γ corresponds to a_5 in G_6):

$$u_x^4 = u_y^2 = u_z^2 = 1, u_yu_x = u_xu_yu_z, [u_x, u_z] = -1,$$
$$[u_y, u_z] = 1, [u_x^2, u_y] = -1, [u_x^2, u_z] = 1.$$

The elements u_x, u_y, u_z change their places with the elements of K/k in this manner: $u_x\sqrt{r\alpha} = x(\sqrt{r\alpha})u_x = \sqrt{r\alpha'}u_x$.

In order to obtain the decomposition, we have to use the well-known theorem: *If A is a central simple finite dimensional algebra over k and B is a subalgebra of A, then $A = B \otimes C_A(B)$, where $C_A(B)$ is the centralizer of B in A.* Calculations show that the following subalgebras Γ_1, Γ_2 and Γ_3 centralize each other:

$$\begin{aligned}
\Gamma_1 &: i_1 = \sqrt{b}, j_1 = u_y;\\
\Gamma_2 &: i_2 = u_x^2\sqrt{b}, j_2 = \sqrt{r\alpha'}[-(\alpha_1+\beta_1) + \alpha_2\sqrt{a} + \beta_2\sqrt{b}u_x^2];\\
\Gamma_3 &: i_3 = u_z\sqrt{a}, j_3 = (\sqrt{s\beta} + \sqrt{s\beta'})[(\beta_1+1) - \sqrt{a} + ((\beta_1-1) - \sqrt{a})u_z],
\end{aligned}$$

where $i_1^2 = b$, $j_1^2 = 1$, $i_1j_1 = -j_1i_1$, $i_2^2 = b$, $j_2^2 = 2a(\alpha_1+\beta_1)r = adr$, $i_2j_2 = -j_2i_2$, $i_3^2 = a$, $j_3^2 = 8s(\beta_1^2 - a) = 8sb\beta_2^2$. Then $\Gamma = \Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3 \otimes \Gamma_4$, where $\Gamma_4 = C_\Gamma(\Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3)$; and

$$[\Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3] = [\Gamma_1][\Gamma_2][\Gamma_3] = (b, 1)(b, adr)(a, 8sb\beta_2^2) = (b, adr)(a, 2sb).$$

We did not succeed in finding explicitly the generators of Γ_4, due to the enormous calculations. Some observations, however, brought us to the supposition that Γ_4 is isomorphic to a quaternion algebra of the type (a, c), where the element

c does not depend on r or s. Fortunately, the same group G_6 is considered in the paper [1] (there denoted as G_1), where is given a Galois extension, realizing the group as a Galois group over arbitrary field of char $\neq 2$. For the most part we try to conform our notations to that paper. In Proposition 3.6 there is proven that $L/k = E(\sqrt{\alpha d}, \sqrt{\beta d}, \sqrt{\gamma})/k$ is a G_6 Galois extension.

Therefore for $r = s = d$ the group G_6 is realizable, so if we assume that c does not depend on r or s, we get

$$[\Gamma] = (b, ad^2)(a, 2db)(a, c) = (b, a)(a, 2db)(a, c) = (a, 2cd) = 1.$$

Then for arbitrary r and s we obtain

$$[\Gamma] = (b, adr)(a, 2sb)(a, c) = (b, dr)(a, ds)(a, 2cd) = (b, dr)(a, ds) \in \mathrm{Br}(k).$$

Now, we will prove that the obstruction is exactly this one by constructing the Galois extensions, realizing G_6.

Theorem 6.1. *The obstruction to solvability of the embedding problem* $(K/k, G_6, \langle a_5\rangle)$*, described above, is* $(b, dr)(a, ds) \in \mathrm{Br}(k)$*. If* $(b, dr)(a, ds) = 1 \in \mathrm{Br}(k)$*, then there exist elements* $\delta_1, \delta_2, \delta_3 \in E$ *and* $v \in k^\times$*, such that* $drv = N_{E/k(\sqrt{a})}(\delta_1), dsv = N_{E/k(\sqrt{b})}(\delta_2), v = N_{E/k(\sqrt{a})}(\delta_3) = N_{E/k(\sqrt{b})}(\delta_3)$*. Then* $M/k = E(\sqrt{r\alpha}, \sqrt{s\beta}, \sqrt{t\delta_1\delta_2\delta_3})/k, t \in k^\times$ *are all Galois extensions, solving the embedding problem* $(K/k, G_6, \langle a_5\rangle)$*.*

Proof. Firstly, assume that $(b, dr)(a, ds) = 1 \in \mathrm{Br}(k)$. By the common slot property (Lemma 5.3) there exists $v \in k^\times$, such that $(b, drv) = (a, dsv) = (ab, v) = 1$. Then there must exist elements $\delta_1, \delta_2, \delta_3 \in E$ and $v \in k^\times$, as in the statement of the theorem. Recall that $N_{E/k(\sqrt{a})}(\gamma) = d\alpha$ and $N_{E/k(\sqrt{b})}(\gamma) = d\beta$. Now, put $\delta = \gamma\delta_1\delta_2\delta_3$. Then

$$N_{E/k(\sqrt{a})}(\delta) = d^2v^2\delta_2^2 r\alpha \in K^{\times 2}$$

and

$$N_{E/k(\sqrt{b})}(\delta) = d^2v^2\delta_1^2 s\beta \in K^{\times 2}.$$

Therefore $M/k = E(\sqrt{r\alpha}, \sqrt{s\beta}, \sqrt{t\delta})/k, t \in k^\times$, is a Galois extension. Now, we shall prove that M/k is a G_6 extension. For convenience, we may assume that $t = 1$. Calculations show that $y\delta/\delta = a_y^2$, for $a_y = dv\sqrt{r\alpha}/(\gamma\delta_1\delta_3)$ and $x\delta/\delta = a_x^2$, for $a_x = dv\sqrt{s\beta}/(\gamma\delta_2\delta_3)$. Therefore $a_y y a_y = 1$, so the preimage of y in $\mathrm{Gal}(M/k)$ is of order 2 and $a_x x a_x x^2 a_x x^3 a_x = 1$, so the preimage of x is of order 4. Denote the preimages of x, y and z by a_1, a_2 and a_3 respectively. Since $z\delta/\delta = [z, x]\delta/\delta = 1$, we obtain that a_3 and $[a_3, a_1]$ are of order 2. Additional calculations show that the remaining relations necessary to have the group G_6 are also fulfilled.

Now, assume that $M/k = E(\sqrt{r\alpha}, \sqrt{s\beta}, \sqrt{\delta'})/k$ is a G_6 extension. According to Theorem 1.1 and some additional checks, involving the relations in the group, we conclude that $\delta' = t\delta_1\delta_2\delta_3$, where $\delta_1 \in k(\sqrt{b}), \delta_2 \in k(\sqrt{a}), \delta_3 \in k(\sqrt{ab})$ and $t \in k^\times$. Then there always exist elements v_1, v_2 and $v_3 \in k^\times$, such that $drv_1 = N_{E/k(\sqrt{a})}(\delta_1), dsv_2 = N_{E/k(\sqrt{b})}(\delta_2)$ and $v_3 = N_{E/k(\sqrt{a})}(\delta_3) = N_{E/k(\sqrt{b})}(\delta_3)$. Since M/k is Galois, we must have $y\delta'/\delta' = a_y^2$, for some $a_y \in k^\times$. But $y\delta'/\delta' = d^2r\alpha v_1v_3/(\gamma^2\delta_1^2\delta_3^2)$, therefore v_1v_3 is in $K^{\times 2} \cap k = k^{\times 2} \cup ak^{\times 2} \cup bk^{\times 2} \cup abk^{\times 2}$. The splitting of the quaternion algebras (a, b) and (a, a) implies that we can reduce the possibilities to this one: $v_1v_3 \in k^{\times 2}$. Thus, we can assume that $v_1 = v_3$. Similarly, from $x\delta'/\delta' = a_x^2$, we obtain that $v_2 = v_3$. By applying the common slot property in the reverse direction, we obtain the obstruction. □

Remark. Another approach for the calculation of the obstruction, valid for another description of $D \curlywedge C$ extensions, can be found in the work [15].

Finally, consider the special case $r = s$. Then $[\Gamma] = (ab, dr)$. Assume that $(ab, dr) = 1$, i.e., $\exists \gamma_1, \gamma_2 \in k$, such that $dr = \gamma_1^2 - ab\gamma_2^2$ and denote $M = E(\sqrt{r\alpha}, \sqrt{r\beta}, \sqrt{(\gamma_1 + \gamma_2\sqrt{ab})\gamma})$. Then M/k is a Galois extension:

$$\begin{aligned}
&(\gamma_1 + \gamma_2\sqrt{ab})\gamma x[(\gamma_1 + \gamma_2\sqrt{ab})\gamma] = \\
&(\gamma_1^2 - ab\gamma_2^2)\gamma x(\gamma) = d^2r\beta \in K^{\times 2}; \\
&(\gamma_1 + \gamma_2\sqrt{ab})\gamma y[(\gamma_1 + \gamma_2\sqrt{ab})\gamma] = d^2r\alpha \in K^{\times 2}.
\end{aligned}$$

Similarly to the proof of the previous Theorem we verify that M/k is a G_6 Galois extension.

7. Embedding problems with cyclic kernel of order 4. The investigation of the groups G_7, G_8, G_{11} and G_{15} requires a different approach. Instead of embedding problems with kernel of order 2, we will discuss embedding problems with cyclic kernel of order 4. The reason is this: for each of these groups the element a_1 is of order 8 and $\langle a_1^2 \rangle$ is a normal cyclic subgroup of order 4.

Now, we write down the criteria from [8]. Let K/k be a finite Galois extension with Galois group F and assume that $i = \sqrt{-1}$ is in K. Also, let

$$1 \to C_4 \to G \underset{\psi}{\to} F \to 1 \tag{7.1}$$

be a group extension. We identify C_4 with the group μ_4 of the fourth roots of unity. We then have two F-module actions on $\mu_4 \cong C_4 :$ the Galois action of F on $\mu_4 \subset K^\times$, which we will write as $(\sigma, \zeta) \mapsto \sigma\zeta$, and the action of F on C_4 induced by (7.1), which we will write as $(\sigma, \zeta) \mapsto {}^\sigma\zeta$. If ${}^\sigma\zeta = \sigma\zeta, \forall \sigma \in F$, the embedding problem $(K/k, G, C_4)$ is called *Brauer.* If the embedding problem $(K/k, G, C_4)$ is not Brauer, it is always possible to reduce the embedding problem to two Brauer problems as is seen in the following theorems.

Theorem 7.1. *Let $i \in K$ and let K^N be the fixed field of $N = \{\sigma \in F \mid \sigma i = {}^\sigma i\}$. Then the embedding problem $(K/k, G, C_4)$ is solvable iff the embedding problems given by K/K^N and*

$$1 \to \mu_4 \to \psi^{-1}(N) \underset{\psi}{\to} N \to 1,$$

respectively by K/k and

$$1 \to \mu_2 \to G/C_2 \underset{\psi'}{\to} F \to 1$$

are solvable.

Theorem 7.2. *Let $i \notin K$. Extend the elements $\sigma \in F$ to $K(i)$ by $\sigma i = i$, and let κ be the generator of $\mathrm{Gal}(K(i)/K)$. Let $N = \{\sigma \in F \mid {}^\sigma i = i\}, K(i)^N = k(\sqrt{b})$, and let $L = k(i\sqrt{b})$. Then $\mathrm{Gal}(K(i)/L) \cong F$ by restriction, and the embedding problem $(K/k, G, C_4)$ is solvable iff the embedding problems given by $K(i)/L$ and*

$$1 \to \mu_4 \to G \underset{\psi}{\to} F \to 1,$$

respectively by K/k and

$$1 \to \mu_2 \to G/\mu_2 \underset{\psi'}{\to} F \to 1$$

are solvable.

8. The group G_7. As we noted in the previous section, $\langle a_4 \rangle$ is a normal subgroup of G_7, and also $G_7/\langle a_4 \rangle \cong D_8$. We have to find the Brauer problem, i.e., to determine the position of i in a D_8 extension K/k, such that the action of D_8 on $\langle a_4 \rangle$ and $\mu_4 \subset K^\times$ is the same. Let $a =_2 -1$ and $b \in k^\times$ be quadratically independent over k, and let K/k be a D_8 extension, described in Section 2. (For example, we have $-b = \alpha_1^2 + \alpha_2^2$ and $\alpha = \alpha_1 - \alpha_2 i$.) In this case the embedding problem given by K/k and the group extension

$$1 \to \langle a_4 \rangle \to G_7 \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma\tau}}{\longrightarrow} D_8 \to 1 \tag{8.1}$$

is Brauer. Indeed, $\tau i = i$, ${}^{\tau}a_4 = a_1 a_4 a_1^{-1} = a_4, \sigma\tau i = -i$, ${}^{\sigma\tau}a_4 = a_2 a_4 a_2^{-1} = a_4 a_5 = a_4^3$.

Denote by $\Gamma = (K, G_7, i)$ the crossed product algebra, related to the group extension (8.1). Let u_1 correspond to τ in Γ, u_2 correspond to $\sigma\tau$, and $u = u_2 u_1$. We then have the relations:

$$u_1^2 = i,\ u_2^2 = 1,\ u^4 = -1;$$
$$u\sqrt{r\alpha} = \sqrt{r\alpha'}u,\ u\sqrt{r\alpha'} = -\sqrt{r\alpha}u,\ ui = -iu,\ u\sqrt{b} = \sqrt{b}u;$$
$$u_1\sqrt{r\alpha} = \sqrt{r\alpha}u_1,\ u_1\sqrt{r\alpha'} = -\sqrt{r\alpha'}u_1,\ u_1 i = iu_1,\ u_1\sqrt{b} = -\sqrt{b}u_1.$$

The relations $a_3 = a_1^2(a_2a_1)^2$ and $a_1^{-1}(a_2a_1)a_1 = (a_2a_1)^3 a_4 a_5$ imply $uu_1 = iu_1u^3$. Calculations show that Γ is decomposed as tensor product of the following three quaternion subalgebras:

$$\begin{aligned} \Gamma_1 &: \ i_1 = \sqrt{b},\ j_1 = (1+i)u_1; \\ \Gamma_2 &: \ i_2 = \sqrt{b}u^2,\ j_2 = i(\sqrt{r\alpha} + \sqrt{r\alpha'}u^2); \\ \Gamma_3 &: \ i_3 = i,\ j_3 = u + u^3. \end{aligned}$$

Therefore $[\Gamma] = (b, -2)(-b, 2\alpha_1 r)(-1, -2) = (-b, -\alpha_1 r)$. We can summarize:

Theorem 8.1. *The obstruction to solvability of the Brauer embedding problem $(K/k, G_7, \langle a_4 \rangle)$, described above, is $(-b, -\alpha_1 r) \in \mathrm{Br}(k)$.*

Note that a necessary condition to solvability of the Brauer problem $(K/k, G_8, \langle a_4 \rangle)$ is the solvability of the associated embedding problem given by K/k and the group extension

$$1 \to \langle a_4 \rangle / \langle a_5 \rangle \to G_7/\langle a_5 \rangle \cong D \curlywedge C \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma\tau}}{\longrightarrow} D_8 \to 1.$$

The obstruction is $(-1, -b)$ which equals 1. This explains why we have obtained the decomposition of Γ so easily. If the obstruction were not trivial, the calculations would have been much complicated. Fortunately, the same situation happens for the groups G_8, G_{11} and G_{15}.

Next, we will discuss in details all five possibilities regarding to the position of i in K/k.

(1) Let $i \in k$ and let K/k be arbitrary D_8 extension. Consider now the embedding problem given by K/k and the group extension (8.1). We then obtain

$$N = \{\rho \in D_8 \mid {}^{\rho}a_4 = \rho i = i\} = \{1, \tau, \sigma^2, \sigma^2\tau\} \cong C_2^2;$$
$$\psi^{-1}(N) = \langle a_1, (a_2a_1)^2\rangle = \langle a_1, a_3\rangle \cong M_{16},$$

where ψ is the homomorphism $G_7 \to D_8$ from the group extension (8.1). Let $E = K^N = k(\sqrt{a})$. Then we can write $K = E(\sqrt{c}, \sqrt{d})$, where $c = r\alpha$ and $d = b$. By Theorem 7.1 the embedding problem $(K/k, G_7, \langle a_4\rangle)$ is reduced to the embedding problem given by K/E and the group extension

$$1 \to \mu_4 \cong \langle a_4\rangle \to \psi^{-1}(N) \cong M_{16} \underset{\psi}{\to} N \cong C_2^2 \to 1,$$

respectively by K/k and

$$1 \to \langle a_4\rangle/\langle a_5\rangle \to G_7/\langle a_5\rangle \cong D \curlywedge C \underset{\substack{a_1\mapsto\tau \\ a_2\mapsto\sigma\tau}}{\longrightarrow} D_8 \to 1.$$

Let $N \cong C_2^2$ be generated by σ_1 and σ_2, such that

$$\sigma_1 \;:\; \sqrt{c} \mapsto -\sqrt{c}, \quad \sqrt{d} \mapsto \sqrt{d};$$
$$\sigma_2 \;:\; \sqrt{c} \mapsto \sqrt{c}, \quad \sqrt{d} \mapsto -\sqrt{d}.$$

In this way the homomorphism $\psi : M_{16} = \langle a_1, a_3\rangle \to N$ is described by $a_1 \mapsto \sigma_2$ and $a_3 \mapsto \sigma_1$. Then the embedding problem $(K/E, M_{16}, \mu_4 = \langle a_4\rangle)$ is solvable iff $(d, d) = (d, 2c)(-1, x) = 1 \in \mathrm{Br}(E)$ for some $x \in E$. But $i \in k$, whence the embedding problem is solvable iff $(d, 2c) = (b, 2r(\alpha_1 - \alpha_2\sqrt{a})) = 1 \in \mathrm{Br}(E)$. Furthermore, the embedding problem $(K/k, D \curlywedge C, \mu_2)$ is solvable iff $(b, -1)(a, -1) = 1$ i.e., $(ab, -1) = 1 \in \mathrm{Br}(k)$, which holds in this case.

(2) $a =_2 -1$. This is the Brauer problem (see Theorem 8.1).

(3) $b =_2 -1$. Here $b = -\beta^2, \beta \in k^\times$ and $ab = -a\beta^2 = \alpha_1 - a\alpha_2^2$ for $\alpha_1 = 0$, $\alpha_2 = -\beta$. For $\alpha = \alpha_1 - \alpha_2\sqrt{a} = \sqrt{\beta^2 a}$, we get $\sqrt{r\alpha} = \sqrt[4]{a'}$, where $a' = r^2\beta^2 a$. Then $K/k = k(\sqrt[4]{a'}, i)/k$ is a D_8 extension. The generators σ and τ of D_8 act, for example, thus:

$$\sigma \;:\; \sqrt[4]{a'} \mapsto \sqrt[4]{a'}i, \quad i \mapsto i;$$
$$\tau \;:\; \sqrt[4]{a'} \mapsto \sqrt[4]{a'}, \quad i \mapsto -i.$$

We have that

$$N = \{1, \sigma^2, \sigma\tau, \sigma^3\tau\} = \{\rho \in D_8 \mid \ {}^{\rho}i = \rho i\}.$$

Put $x = (a_2a_1)^2 = a_3a_4^{-1}, y = a_2, z = a_3$ and $E = K^N = k(\sqrt[4]{a'}i)$. Then we have $\psi^{-1}(N) = \langle x, y, a_4\rangle = \langle x, y\rangle \times \langle z\rangle \cong D_8 \times C_2$ and $K = E(\sqrt{c}, \sqrt{d})$, where $c = 2\sqrt{a'}i, d = -1$. The group $N \cong C_2^2$ is generated by elements σ_1 and σ_2, such that

$$\begin{array}{lcl}\sigma_1 & : & \sqrt{c} \mapsto -\sqrt{c}, \quad \sqrt{d} \mapsto \sqrt{d};\\ \sigma_2 & : & \sqrt{c} \mapsto \sqrt{c}, \quad \sqrt{d} \mapsto -\sqrt{d}.\end{array}$$

The homomorphism $\psi : D_8 \times C_2 \to C_2^2$ can be described in this way: $x \mapsto \sigma_1, y \mapsto \sigma_2, z \mapsto \sigma_1$, since $z = a_4x$.

Consider now the associated embedding problem given by K/E and the group extension

$$1 \to \mu_4 \cong \langle a_4\rangle \to \psi^{-1}(N) \cong D_8 \times C_2 \underset{\psi}{\to} N \cong C_2^2 \to 1.$$

A necessary condition to solvability of the latter embedding problem is the solvability of the associated embedding problem given by K/E and

$$1 \to \mu_2 \cong \langle a_4\rangle/\langle a_5\rangle \to \psi^{-1}(N)/\langle a_5\rangle \cong C_2^3 \underset{\psi}{\to} N \cong C_2^2 \to 1.$$

The associated embedding problem is solvable iff $\exists e \in E^\times$, such that c, d and e are quadratically independent over E. Denote $K_1 = E(\sqrt{c}, \sqrt{d}, \sqrt{e})$ and let us consider the embedding problem given by K_1/E and

$$1 \to \mu_2 \cong \langle a_5\rangle \to \psi^{-1}(N) \cong D_8 \times C_2 \underset{\substack{x \mapsto \sigma_1\\ y \mapsto \sigma_2\\ a_4 \mapsto \sigma_3}}{\longrightarrow} C_2^3 \to 1.$$

We have the relations $x^2 = -1, y^2 = 1, a_4^2 = -1, xy = -yx, a_4x = xa_4$ and $a_4y = -ya_4$. From Theorem 2.1 follows that the latter embedding problem is solvable iff $(c, c)(c, d)(e, e)(d, e) = 1 \in \mathrm{Br}(E)$, which, clearly, always holds (here $d = -1$).

The embedding problem given by K/k and the group extension

$$1 \to \langle a_4\rangle/\langle a_5\rangle \to G_7/\langle a_5\rangle \cong D \leftthreetimes C \underset{\substack{a_1 \mapsto \tau\\ a_2 \mapsto \sigma\tau}}{\to} D_8 \to 1$$

is solvable iff $(-a,-1)=1\in \mathrm{Br}(k)$.

Thus we can summarize: The embedding problem $(K/k, G_7, \langle a_4\rangle)$ is solvable iff $\exists e\in E^\times$, such that $c=2\sqrt{a'}i$, $d=-1$ and e are quadratically independent over $E=k(\sqrt[4]{a'}i)$ and $(-a,-1)=1\in \mathrm{Br}(k)$.

(4) $ab =_2 -1$. Here $\sigma(i)=-i$ and $\tau(i)=-i$, whence $N=\langle\sigma\rangle$. Let K/k be a D_8 extension. Denote $E=K^N=k(\sqrt{b})=k(\sqrt{a}i)$. Put $x=a_2a_1$ and $y=a_3$. Then $\psi^{-1}(N)=\langle x,y\rangle\cong M_{16}$ and $\psi^{-1}(N)/\langle a_5\rangle\cong C_4\times C_2$. The embedding problem $(K/E, C_4\times C_2,\mu_2)$ is solvable iff $\exists e\in E^\times$, such that a and e are quadratically independent over E. Denote $K_1=K(\sqrt{e})$ and consider the embedding problem given by K_1/E and the group extension

$$1\to \mu_2\cong\langle a_5\rangle\to\psi^{-1}(N)\underset{\substack{x\mapsto\rho_1\\ y\mapsto\rho_2}}{\longrightarrow} C_4\times C_2\to 1.$$

Here we need some preparation before applying Theorem 2.2. Let $\beta_1=r\alpha_1$, $\beta_2=r\alpha_2$ and $ab=-\beta^2$, where $\beta\in k^\times$. Then $\beta_1^2-a\beta_2^2=r^2ab=-r^2\beta^2$. For $\gamma=ir/\sqrt{a}\in E$ we get $a\gamma^2=-r^2$. From [2] now follows that the embedding problem is solvable iff $(a,2e)(-1,\beta_1)=(a,2r\alpha_1e)=1\in\mathrm{Br}(E)$, since $a =_2 -1 \pmod{E^2}$.

Therefore the embedding problem $(K/E,\psi^{-1}(N),\langle a_4\rangle)$ is solvable iff $\exists e\in E^\times$, such that $(a,2r\alpha_1e)=1\in\mathrm{Br}(E)$. The associated embedding problem $(K/k, G_7/\langle a_5\rangle,\mu_2)$ is solvable iff $(-1,-1)=1\in\mathrm{Br}(k)$. Whence the embedding problem $(K/k,G_7,\langle a_4\rangle)$ is solvable iff $\exists e\in E^\times$, such that $(a,2r\alpha_1e)=1\in\mathrm{Br}(E)$ and $(-1,-1)=1\in\mathrm{Br}(k)$, where a and e are quadratically independent over E.

Finally,

(5) $i\notin K/k$ – arbitrary D_8 extension. Let κ generate $\mathrm{Gal}(K(i)/K)$ and identify $\mathrm{Gal}(K/k)$ with $\mathrm{Gal}(K(i)/k(i))$. According to Theorem 7.2 we must take the group $N=\langle\sigma\kappa,\tau\rangle$, which is the Galois group of $K(i)/k(i\sqrt{a})$. Then the embedding problem given by $K(i)/k(i\sqrt{a})$ and

$$1\to\langle a_4\rangle\to G_7\underset{\substack{a_1\mapsto\tau\\ a_2\mapsto\sigma\kappa\tau}}{\longrightarrow} D_8\to 1$$

is Brauer. Since $-a\in k(i\sqrt{a})^2$, we get $\alpha=\alpha_1-\alpha_2\sqrt{a}=\alpha_1-i\alpha_2'$ for proper $\alpha_2'\in k(i\sqrt{a})^\times$. Thus we obtain that the embedding problem $(K(i)/k(i\sqrt{a}),G_7,\langle a_4\rangle)$ is solvable iff $(-b,-\alpha_1r)=1\in\mathrm{Br}(k(i\sqrt{a}))$. It remains only to add the condition $(ab,-1)=1\in\mathrm{Br}(k)$ to solvability of the associated embedding problem $(K/k,G_7/\langle a_5\rangle,\langle a_5\rangle)$.

9. The group G_8. Here again $\langle a_4\rangle$ is a normal subgroup of G_8, and $G_8/\langle a_4\rangle \cong D_8$. Let $a =_2 -1$ and $b \in k^\times$ be quadratically independent over k, and let K/k be a D_8 extension, described in Section 2. We then have that the embedding problem given by K/k and the group extension

$$1 \to \langle a_4\rangle \to G_8 \underset{\substack{a_1\mapsto\tau\\ a_2\mapsto\sigma\tau}}{\longrightarrow} D_8 \to 1 \tag{9.1}$$

is Brauer. Let $\Gamma = (K, G_8, i)$ be the crossed product algebra, related to the group extension (9.1). Let u_1 correspond to τ in Γ, u_2 correspond to $\sigma\tau$, and $u = u_2u_1$. Then we have the relations $u_1^2 = i, u_2^2 = -1, u^4 = -1$ and $uu_1 = -iu_1u^3$. The algebra Γ is decomposed as tensor product of the following three quaternion subalgebras:

$$\begin{aligned}\Gamma_1 &: \ i_1 = \sqrt{b},\ j_1 = (1-i)u_1;\\ \Gamma_2 &: \ i_2 = \sqrt{b}u^2,\ j_2 = i(\sqrt{r\alpha} + \sqrt{r\alpha'}u^2);\\ \Gamma_3 &: \ i_3 = i,\ j_3 = u + u^3.\end{aligned}$$

Therefore $[\Gamma] = (b, 2)(-b, 2\alpha_1 r)(-1, -2) = (-b, \alpha_1 r)(-1, -1)$. We can summarize:

Theorem 9.1. *The obstruction to solvability of the Brauer embedding problem $(K/k, G_8, \langle a_4\rangle)$, described above, is $(-b, \alpha_1 r)(-1, -1) \in \mathrm{Br}(k)$.*

The associated embedding problem $(K/k, G_8/\langle a_5\rangle, \langle a_5\rangle)$ is always solvable. We will again discuss all 5 cases. Our goal is to prove the automatic realizability $G_8 \Rightarrow G_7$.

(1) $i \in k$. The embedding problem $(K/k, G_8, \langle a_4\rangle)$ is solvable iff $(d, 2c) = 1 \in \mathrm{Br}(E)$, i.e., $(b, 2r(\alpha_1 - \alpha_2\sqrt{a})) = 1 \in \mathrm{Br}(E)$. Therefore in this case we obtain the automatic realizability $G_7 \Leftrightarrow G_8$.

(2) $a =_2 -1$. This is the Brauer problem. If we replace r by $-\alpha_1$ in the obstruction $(-b, -\alpha_1 r)$ to solvability of the embedding problem $(K/k, G_7, \mu_4)$, we obtain the automatic realizability $G_8 \Rightarrow G_7$.

(3) $b =_2 -1$. We keep the notations of the same case for the group G_7. Consider the embedding problem given by K/k and the group extension (9.1). Again, $N = \{1, \sigma^2, \sigma\tau, \sigma^3\tau\}$. Put $x = (a_2a_1)^2 = a_3a_4, y = a_2$ and $z = a_3$. Then $\psi^{-1}(N) = \langle x, y, a_4\rangle = \langle x, y\rangle \times \langle z\rangle \cong Q_8 \times C_2$.

Since $z = xa_4^{-1}$, the homomorphism $\psi : Q_8 \times C_2 \to C_2^2$ can be described by $x \mapsto \sigma_1, y \mapsto \sigma_2, z \mapsto \sigma_1$. Consider the associated embedding problem given by K/E and the group extension

$$1 \to \mu_4 \cong \langle a_4 \rangle \to \psi^{-1}(N) \cong Q_8 \times C_2 \underset{\psi}{\to} N \cong C_2^2 \to 1.$$

A necessary condition to solvability of the latter embedding problem is the solvability of the associated embedding problem given by K/E and

$$1 \to \mu_2 \cong \langle a_4 \rangle / \langle a_5 \rangle \to \psi^{-1}(N)/\langle a_5 \rangle \cong C_2^3 \underset{\psi}{\to} N \cong C_2^2 \to 1.$$

The latter associated embedding problem is solvable iff $\exists e \in E^\times$, such that c, d and e are quadratically independent over E. Denote $K_1 = E(\sqrt{c}, \sqrt{d}, \sqrt{e})$ and let us consider the embedding problem given by K_1/E and

$$1 \to \mu_2 \cong \langle a_5 \rangle \to \psi^{-1}(N) \cong Q_8 \times C_2 \underset{\substack{x \mapsto \sigma_1 \\ y \mapsto \sigma_2 \\ a_4 \mapsto \sigma_3}}{\longrightarrow} C_2^3 \to 1.$$

We have the relations $x^2 = y^2 = a_4^2 = -1$, $xy = -yx$, $a_4 x = x a_4$ and $a_4 y = y a_4$. From Theorem 2.1 follows that the latter embedding problem is solvable iff $(c,c)(d,d)(e,e)(c,d)(d,e) = 1 \in \mathrm{Br}(E) \iff (-1,-1) = 1 \in \mathrm{Br}(E)$.

The embedding problem given by K/k and the group extension

$$1 \to \langle a_4 \rangle / \langle a_5 \rangle \to G_8/\langle a_5 \rangle \cong D \curlywedge C \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma\tau}}{\to} D_8 \to 1$$

is solvable iff $(-a,-1) = 1 \in \mathrm{Br}(k)$.

Thus we can summarize: The embedding problem $(K/k, G_8, \langle a_4 \rangle)$ is solvable iff $\exists e \in E^\times$, such that $c = 2\sqrt{a'}i, d = -1$ and e are quadratically independent over $E = k(\sqrt[4]{a'}i), (-1,-1) = 1 \in \mathrm{Br}(E)$ and $(-a,-1) = 1 \in \mathrm{Br}(k)$. In particular, we obtain again the automatic realizability $G_8 \Rightarrow G_7$.

(4) Analogously to the group G_7, we obtain that the embedding problem $(K/k, G_8, \langle a_4 \rangle)$ is solvable iff $(a, 2r\alpha_1 e) = 1 \in \mathrm{Br}(E)$ and $(-1,-1) = 1 \in \mathrm{Br}(k)$.

Finally,

(5) a, b and -1 are quadratically independent over k. Similarly to the group G_7 we obtain that the embedding problem$(K/k, G_8, \langle a_4 \rangle)$ is solvable iff $(-b, \alpha_1 r)(-1,-1) = 1 \in \mathrm{Br}(k(i\sqrt{a}))$ and $(-1, ab) = 1 \in \mathrm{Br}(k)$.

Comparing each case for the groups G_7 and G_8 we have shown in this way that the automatic realizability $G_8 \Rightarrow G_7$ holds.

10. The group G_{11}. We did not discover an automatic realizability between the groups G_{11} and G_{15}, so we decided to investigate only the Brauer problem. Again, the centre $Z(G_{11}) = \langle a_4 \rangle$ is isomorphic to C_4 and the quotient group $G_{11}/\langle a_4 \rangle$ is isomorphic to D_8. Let K/k be a D_8 extension, and let $i \in k$. Then the embedding problem given by K/k and the group extension

$$1 \to \langle a_4 \rangle \to G_{11} \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma\tau}}{\longrightarrow} D_8 \to 1 \tag{10.1}$$

is Brauer. Indeed, $\tau i = i$, ${}^{\tau}a_4 = a_1 a_4 a_1^{-1} = a_4, \sigma\tau i = i$, ${}^{\sigma\tau}a_4 = a_2 a_4 a_2^{-1} = a_4$. Denote by $\Gamma = (K, G_{11}, i)$ the crossed product algebra, related to the group extension (10.1). Let u_1 correspond to τ in Γ, u_2 correspond to $\sigma\tau$, and $u = u_2 u_1$. We then have the relations:

$$u_1^2 = i,\ u_2^2 = 1,\ u^4 = 1,\ uu_1 = iu_1u^3;$$
$$u\sqrt{r\alpha} = \sqrt{r\alpha'}u,\ u\sqrt{r\alpha'} = -\sqrt{r\alpha}u,\ u\sqrt{a} = -\sqrt{a}u,\ u\sqrt{b} = \sqrt{b}u;$$
$$u_1\sqrt{r\alpha} = \sqrt{r\alpha}u_1,\ u_1\sqrt{r\alpha'} = -\sqrt{r\alpha'}u_1,\ u_1\sqrt{a} = \sqrt{a}u_1,\ u_1\sqrt{b} = -\sqrt{b}u_1.$$

Calculations show that Γ is decomposed as tensor product of the following three quaternion subalgebras:

$$\begin{aligned}
\Gamma_1 &: i_1 = \sqrt{b},\ j_1 = u_1;\\
\Gamma_2 &: i_2 = \sqrt{b}u^2,\ j_2 = \sqrt{a}[\sqrt{r\alpha} + i\sqrt{r\alpha'}u^2];\\
\Gamma_3 &: i_3 = \sqrt{a},\ j_3 = u + iu^3.
\end{aligned}$$

Therefore $[\Gamma] = (b, i)(b, 2\alpha_1 ra)(a, 2i) = (b, \alpha_1 r) \in \mathrm{Br}(k)$, since $2i = (1+i)^2 \in k^2$ and $(a, -b) = 1 \in \mathrm{Br}(k)$. We can summarize:

Theorem 10.1. *The obstruction to solvability of the Brauer embedding problem $(K/k, G_{11}, \langle a_4 \rangle)$, described above, is $(b, \alpha_1 r) \in \mathrm{Br}(k)$.*

In particular, if $i \in k$, we obtain the automatic realizability $D_8 \Rightarrow G_{11}$ (we have to replace r by α_1 in the obstruction). Here again the associated embedding problem $(K/k, G_{11}/\langle a_5 \rangle \cong D \curlywedge C, \langle a_5 \rangle)$ is always solvable.

11. The group G_{15}. As before, the centre $Z(G_{15}) = \langle a_4 \rangle$ is isomorphic to C_4 and the quotient group $G_{15}/\langle a_4 \rangle$ is isomorphic to D_8. Let K/k be a D_8

extension, and let $i \in k$. Then the embedding problem given by K/k and the group extension

$$1 \to \langle a_4 \rangle \to G_{15} \underset{\substack{a_1 \mapsto \tau \\ a_2 \mapsto \sigma}}{\longrightarrow} D_8 \to 1 \tag{11.1}$$

is Brauer. Denote by $\Gamma = (K, G_{15}, i)$ the crossed product algebra, related to the group extension (11.1). Let u_1 correspond to τ in Γ and u_2 correspond to σ. We then have the relations:

$$u_1^2 = i,\ u_2^4 = -1,\ u_1u_2 = -u_2^3u_1;$$
$$u_1\sqrt{r\alpha} = \sqrt{r\alpha}u_1,\ u_1\sqrt{r\alpha'} = -\sqrt{r\alpha'}u_1,\ u_1\sqrt{a} = \sqrt{a}u_1,\ u_1\sqrt{b} = -\sqrt{b}u_1;$$
$$u_2\sqrt{r\alpha} = \sqrt{r\alpha'}u_2,\ u_2\sqrt{r\alpha'} = -\sqrt{r\alpha}u_2,\ u_2\sqrt{a} = -\sqrt{a}u_2,\ u_2\sqrt{b} = \sqrt{b}u_2.$$

The algebra Γ is decomposed as tensor product of the following three quaternion subalgebras:

$$\begin{aligned} \Gamma_1 &: \ i_1 = \sqrt{b},\ j_1 = \sqrt{a}u_1; \\ \Gamma_2 &: \ i_2 = \sqrt{b}u_2^2,\ j_2 = \sqrt{a}[\sqrt{r\alpha} + \sqrt{r\alpha'}u_2^2]; \\ \Gamma_3 &: \ i_3 = \sqrt{a},\ j_3 = u_2 + u_2^3. \end{aligned}$$

Therefore $[\Gamma] = (b, ai)(-b, 2\alpha_1 ra)(a, -2) = (b, \alpha_1 r)(a, 2) \in \mathrm{Br}(k)$, since $2i = (1+i)^2 \in k^2$ and $(a, -b) = 1 \in \mathrm{Br}(k)$. We can summarize:

Theorem 11.1. *The obstruction to solvability of the Brauer embedding problem* $(K/k, G_{15}, \langle a_4 \rangle)$*, described above, is* $(b, \alpha_1 r)(a, 2) \in \mathrm{Br}(k)$.

Here again the associated embedding problem $(K/k, G_{15}/\langle a_5 \rangle \cong Q \curlywedge C, \langle a_5 \rangle)$ is always solvable.

12. The groups $G_{17}, G_{18}, G_{19}, G_{20}, G_{49}$ and G_{50}. We decided to include in this section the main obstructions to realizability of these groups for convenience of the reader. The non abelian groups of exponent 16 are: G_{17} – the modular group M_{32}, G_{18} – the dihedral group D_{32}, G_{19} – the semidihedral group SD_{32}, and G_{20} – the quaternion group Q_{32}. We begin by giving the obstructions to solvability of the Brauer problems for these groups found in [8], [11] and [12].

Let the group G be generated by two elements s and t, such that s is of order 16. Identify the cyclic subgroup $\langle s^4 \rangle$ with the group μ_4 of the fourth roots of unity, i.e., $s^4 = i, s^8 = -1$. Let also $t^2 = \varepsilon_1$ and $ts = \varepsilon_2 s^{-1} t$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Then we have the isomorphisms: for $\varepsilon_1 = \varepsilon_2 = 1, G \cong D_{32}$; for

$\varepsilon_1 = 1, \varepsilon_2 = -1, G \cong SD_{32}$; for $\varepsilon_1 = -1, \varepsilon_2 = 1, G \cong Q_{32}$. Let $K/k = k(\sqrt[4]{a}, i)/k$, where a and -1 are quadratically independent over k. Then the embedding problem given by the D_8 extension K/k and the group extension

$$1 \to \mu_4 \underset{i \mapsto s^4}{\longrightarrow} G \underset{\substack{s \mapsto \sigma \\ t \mapsto \tau}}{\longrightarrow} D_8 \to 1$$

is Brauer. The obstruction to solvability of this embedding problem is

$$(-1, \varepsilon_1)(2, \alpha_1\beta_1)(a, \varepsilon_2\alpha_1(\alpha_1 - 1)) \in \mathrm{Br}(k),$$

where $\alpha_1 \in k^\times, \beta_1 \in k$ are such that $\alpha_1^2 + a\beta_1^2 = 2$. For the remaining cases see [8].

Now, let the modular group M_{32} be generated by elements s and t, such that $s^{16} = t^2 = 1, ts = s^9 t$. Let $i \in k$, and let $K/k = k(\sqrt[4]{a}, \sqrt{b})/k$, where a and b are quadratically independent over k. Assume the group $C_4 \times C_2$ is generated by elements ρ_1 and ρ_2, which act on K/k thus:

$$\begin{aligned} \rho_1 &: \sqrt[4]{a} \mapsto \sqrt[4]{a}i, \ \sqrt{b} \mapsto \sqrt{b}; \\ \rho_2 &: \sqrt[4]{a} \mapsto \sqrt[4]{a}, \ \sqrt{b} \mapsto -\sqrt{b}. \end{aligned}$$

Then the embedding problem given by K/k and the group extension

$$1 \to \mu_4 \underset{i \mapsto s^4}{\longrightarrow} M_{32} \underset{\substack{s \mapsto \rho_1 \\ t \mapsto \rho_2}}{\longrightarrow} C_4 \times C_2 \to 1$$

is Brauer. The obstruction is $(a, \alpha b)(i, \alpha\beta) \in \mathrm{Br}(k)$, where is necessary that $\exists \alpha \in k^\times, \beta \in k$, such that $\alpha^2 - a\beta^2 = i$. For the remaining cases see [12].

Finally, for the two extra-special groups $DD \cong G_{49}$ and $DQ \cong G_{50}$ we have from [14]:

Proposition 12.1. *There exists a Galois extension L/k with Galois group $\mathrm{Gal}(L/k) \cong DD$ iff there exist $a, b, c, d \in k^\times$, quadratically independent over k, such that $(a, b)(c, d) = 1 \in \mathrm{Br}(k)$.*

Proposition 12.2. *There exists a Galois extension L/k with Galois group $\mathrm{Gal}(L/k) \cong DQ$ iff there exist $a, b, c, d \in k^\times$, quadratically independent over k, such that $(-a, -b)(-1, -1)(c, d) = 1 \in \mathrm{Br}(k)$.*

Appendix

Group	Relations	Centre	Rank	Exp
G_1	$a_1^2a_2^{-1}$, $a_2^2a_3^{-1}$, $a_3^2a_4^{-1}$, $a_4^2a_5^{-1}$, a_5^2	G_1	1	32
G_2	$a_1^2a_4^{-1}$, $a_2^2a_5^{-1}$, $[a_2,a_1]a_3^{-1}$, a_3^2, a_4^2, a_5^2	$\langle a_3,a_4,a_5\rangle$	2	4
G_3	$a_1^2a_3^{-1}$, $a_2^2a_4^{-1}$, $a_3^2a_5^{-1}$, a_4^2, a_5^2	G_3	2	8
G_4	$a_1^2a_3^{-1}$, $a_2^2a_4^{-1}$, $[a_2,a_1]a_5^{-1}$, $a_3^2a_5^{-1}$, a_4^2, a_5^2	$\langle a_3,a_4,a_5\rangle$	2	8
G_5	$a_1^2a_4^{-1}$, a_2^2, $[a_2,a_1]a_3^{-1}$, a_3^2, $a_4^2a_5^{-1}$, a_5^2	$\langle a_3,a_4,a_5\rangle$	2	8
G_6	$a_1^2a_4^{-1}$, a_2^2, $[a_2,a_1]a_3^{-1}$, a_3^2, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, a_4^2, $[a_4,a_1]$, $[a_4,a_2]a_5^{-1}$, $[a_4,a_3]$, a_5^2	$\langle a_5\rangle$	2	8
G_7	$a_1^2a_4^{-1}$, a_2^2, $[a_2,a_1]a_3^{-1}$, a_3^2, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, $a_4^2a_5^{-1}$ $[a_4,a_1]$, $[a_4,a_2]a_5^{-1}$, $[a_4,a_3]$, a_5^2	$\langle a_5\rangle$	2	8
G_8	$a_1^2a_4^{-1}$, $a_2^2a_5^{-1}$, $[a_2,a_1]a_3^{-1}$, a_3^2, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, $a_4^2a_5^{-1}$, $[a_4,a_1]$, $[a_4,a_2]a_5^{-1}$, $[a_4,a_3]$, a_5^2	$\langle a_5\rangle$	2	8
G_9	$a_1^2a_4^{-1}$, a_2^2, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, a_4^2, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{10}	$a_1^2a_4^{-1}$, $a_2^2a_5^{-1}$, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]a_5^{-1}$, a_4^2, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{11}	$a_1^2a_4^{-1}$, a_2^2, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]a_5^{-1}$, $a_4^2a_5^{-1}$, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{12}	$a_1^2a_4^{-1}$, $a_2^2a_3^{-1}$, $[a_2,a_1]a_3^{-1}$, a_3^2, $a_4^2a_5^{-1}$, a_5^2	$\langle a_3,a_4,a_5\rangle$	2	8
G_{13}	$a_1^2a_4^{-1}$, $a_2^2a_3^{-1}$, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, a_4^2, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{14}	$a_1^2a_4^{-1}$, $a_2^2a_5^{-1}a_3^{-1}$, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, a_4^2, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{15}	$a_1^2a_4^{-1}$, $a_2^2a_3^{-1}$, $[a_2,a_1]a_3^{-1}$, $a_3^2a_5^{-1}$, $[a_3,a_1]a_5^{-1}$, $[a_3,a_2]$, $a_4^2a_5^{-1}$, a_5^2	$\langle a_4,a_5\rangle$	2	8
G_{16}	$a_1^2a_3^{-1}$, a_2^2, $a_3^2a_4^{-1}$, $a_4^2a_5^{-1}$, a_5^2	G_{16}	2	16

G_{17}	$a_1^2a_3^{-1}, a_2^2, [a_2, a_1]a_5^{-1}, a_3^2a_4^{-1}, a_4^2a_5^{-1}, a_5^2$	$\langle a_3, a_4, a_5\rangle$	2	16
G_{18}	$a_1^2, a_2^2, [a_2, a_1]a_3^{-1}, a_3^2a_5^{-1}a_4^{-1}, [a_3, a_1]a_4^{-1}, [a_3, a_2]a_4^{-1}, a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, [a_4, a_3], a_5^2$	$\langle a_5\rangle$	2	16
G_{19}	$a_1^2a_5^{-1}, a_2^2, [a_2, a_1]a_3^{-1}, a_3^2a_5^{-1}a_4^{-1}, [a_3, a_1]a_4^{-1}, [a_3, a_2]a_4^{-1}, a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1},]a_4, a_3], a_5^2$	$\langle a_5\rangle$	2	16
G_{20}	$a_1^2a_5^{-1}, a_2^2a_5^{-1}, [a_2, a_1]a_3^{-1}, a_3^2a_5^{-1}a_4^{-1}, [a_3, a_1]a_4^{-1}, [a_3, a_2]a_4^{-1}, a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, [a_4, a_3], a_5^2$	$\langle a_5\rangle$	2	16
G_{21}	$a_1^2a_4^{-1}, a_2^2a_5^{-1}, a_3^2, a_4^2, a_5^2$	G_{21}	3	4
G_{22}	$a_1^2a_5^{-1}, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	4
G_{23}	$a_1^2a_5^{-1}, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	4
G_{24}	$a_1^2a_5^{-1}, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2a_4^{-1}, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	4
G_{25}	$a_1^2a_5^{-1}, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2a_5^{-1}, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	4
G_{26}	$a_1^2a_5^{-1}, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_5^{-1}a_4^{-1}, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	4
G_{27}	$a_1^2, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{28}	$a_1^2, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{29}	$a_1^2a_4^{-1}, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{30}	$a_1^2, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2a_4^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{31}	$a_1^2, a_2^2a_5^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_4^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{32}	$a_1^2a_4^{-1}, a_2^2a_5^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_4^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{33}	$a_1^2, a_2^2a_5^{-1}a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_4^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{34}	$a_1^2, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_5^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4
G_{35}	$a_1^2a_4^{-1}, a_2^2a_4^{-1}, [a_2, a_1]a_4^{-1}, a_3^2a_5^{-1}, [a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2, a_5^2$	$\langle a_4, a_5\rangle$	3	4

G_{36}	$a_1^2a_4^{-1}, a_2^2, a_3^2, a_4^2a_5^{-1}, a_5^2$	G_{36}	3	8
G_{37}	$a_1^2a_4^{-1}, a_2^2, [a_2, a_1]a_5^{-1}, a_3^2,$ $a_4^2a_5^{-1}, a_5^2$	$\langle a_3, a_4, a_5\rangle$	3	8
G_{38}	$a_1^2a_4^{-1}, a_2^2, a_3^2, [a_3, a_2]a_5^{-1},$ $a_4^2a_5^{-1}, a_5^2$	$\langle a_1, a_4, a_5\rangle$	3	8
G_{39}	$a_1^2, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2, a_4^2a_5^{-1},$ $[a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, a_5^2$	$\langle a_3, a_5\rangle$	3	8
G_{40}	$a_1^2a_5^{-1}, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2, a_4^2a_5^{-1},$ $[a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, a_5^2$	$\langle a_3, a_5\rangle$	3	8
G_{41}	$a_1^2a_5^{-1}, a_2^2a_5^{-1}, [a_2, a_1]a_4^{-1}, a_3^2,$ $a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, a_5^2$	$\langle a_3, a_5\rangle$	3	8
G_{42}	$a_1^2, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2a_5^{-1},$ $a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, a_5^2$	$\langle a_3, a_5\rangle$	3	8
G_{43}	$a_1^2, a_2^2, [a_2, a_1]a_4^{-1}, a_3^2, [a_3, a_1]a_5^{-1},$ $[a_3, a_2], a_4^2a_5^{-1}, [a_4, a_1]a_5^{-1},$ $[a_4, a_2]a_5^{-1}, [a_4, a_3], a_5^2$	$\langle a_5\rangle$	3	8
G_{44}	$a_1^2, a_2^2a_5^{-1}, [a_2, a_1]a_4^{-1}, a_3^2,$ $[a_3, a_1]a_5^{-1}, [a_3, a_2], a_4^2a_5^{-1},$ $[a_4, a_1]a_5^{-1}, [a_4, a_2]a_5^{-1}, [a_4, a_3], a_5^2$	$\langle a_5\rangle$	3	8
G_{45}	$a_1^2a_5^{-1}, a_2^2, a_3^2, a_4^2, a_5^2$	G_{45}	4	4
G_{46}	$a_1^2, a_2^2, [a_2, a_1]a_5^{-1}, a_3^2, a_4^2, a_5^2$	$\langle a_3, a_4, a_5\rangle$	4	4
G_{47}	$a_1^2a_5^{-1}, a_2^2a_5^{-1}, [a_2, a_1]a_5^{-1}, a_3^2,$ a_4^2, a_5^2	$\langle a_3, a_4, a_5\rangle$	4	4
G_{48}	$a_1^2, a_2^2, [a_2, a_1]a_5^{-1}, a_3^2a_5^{-1},$ a_4^2, a_5^2	$\langle a_3, a_4, a_5\rangle$	4	4
G_{49}	$a_1^2, a_2^2, [a_2, a_1]a_5^{-1}, a_3^2,$ $[a_3, a_1], [a_3, a_2]a_5^{-1}, a_4^2,$ $[a_4, a_1]a_5^{-1}, [a_4, a_2], [a_4, a_3], a_5^2$	$\langle a_5\rangle$	4	4
G_{50}	$a_1^2, a_2^2a_5^{-1}, [a_2, a_1]a_5^{-1}, a_3^2a_5^{-1},$ $[a_3, a_1], [a_3, a_2]a_5^{-1}, a_4^2,$ $[a_4, a_1]a_5^{-1}, [a_4, a_2], [a_4, a_3], a_5^2$	$\langle a_5\rangle$	4	4
G_{51}	$a_1^2, a_2^2, a_3^2, a_4^2, a_5^2$	G_{51}	5	2

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