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Embedding Obstructions for the Dihedral, Semidihedral, and Quaternion 2-Groups

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For each of the dihedral, semidihedral, and quaternion 2-groups, we represent the obstructions to certain Brauer problems as tensor products of quaternion algebras. Then we reduce various embedding problems with cyclic 2-kernels into two Brauer problems, thus finding the obstructions in some specific cases.

1. INTRODUCTION

Let K/k be a Galois extension with Galois group F, and let

$$(*) \qquad 1 \to A \to G \underset{\pi}{\to} F \to 1$$

be a finite group extension. The embedding problem $(K/k, G, A)$ then consists of determining whether there exists a Galois extension L/k such that $K \subset L$, $G \cong \mathrm{Gal}(L/k)$, and the homomorphism of restriction to K of the automorphisms from G coincides with π. The group A is called the *kernel* of the embedding problem. If there exists a Galois algebra with the aforementioned properties, then we also talk about "weak" solvability. Given that A is contained in the Frattini subgroup of G, i.e., $\mathrm{rank}(G) = \mathrm{rank}(F)$, the two terms are equivalent.

Let A be a cyclic group of order m, let $\zeta \in K$ be a primitive mth root of unity, and denote $\mu_m = \langle \zeta \rangle \subset K^*$. If F acts on A and μ_m in the same way, then the embedding problem $(K/k, G, A)$ is called a Brauer problem. We can identify A with μ_m and denote by c the 2-coclass of the extension $(*)$ in $H^2(F, \mu_m)$. It is well known (see [Mi2, ILF]) that $(K/k, G, A)$ is

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solvable if and only if c maps to 1 under the map $H^2(F, \mu_m) \to H^2(F, K^*)$, induced by the inclusion $\mu_m \subset K^*$. In this way we consider c as an element of the relative Brauer group $\mathrm{Br}(K/k) \cong H^2(F, K^*)$. The element c is called the (embedding) obstruction to the embedding problem $(K/k, G, A)$. Then $c = 1 \in \mathrm{Br}(k)$ gives us the condition for solvability.

Let k be of characteristic $\neq 2$ and let $m = 2^n$; i.e., let $\zeta \in K$ be a primitive 2^nth root of unity and $A \cong C_{2^n}$. Then we can split the algebra representing the obstruction to the Brauer problem into a tensor product of quaternion algebras and matrix algebras. Namely, for the solvability of the embedding problem $(K/k, G, \mu_{2^n})$ it is necessary the solvability of the associated problem $(K/k, G/C_2, \mu_{2^{n-1}})$. The latter has an obstruction $c^2 \in \mathrm{Br}(K/k)$. Given $c^2 = 1 \in \mathrm{Br}(k)$ by Merkurjev theorem [Me] c can be split into a product of quaternion classes. All needed information about quaternion algebras and Brauer groups can be found in, for example, [La].

If the embedding problem $(K/k, G, C_{2^n})$ is not Brauer but $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, then we can reduce it to two Brauer problems. We do this in Section 2.

In Section 3 we apply Theorem 2.1 and Corollary 2.2 to find the obstruction to the embedding problem given by a D_8 extension K/k and the group extension

$$1 \to C_{2^n} \to G \to D_8 \to 1,$$

where G is isomorphic to either the dihedral, the semidihedral, or the quaternion groups of order 2^{n+3} $(n \geq 1)$. We investigate four such embedding problems in all possible cases according to the location of ζ in K.

The representation of obstructions as products of quaternion classes is a difficult problem even if G is a group of order 16 (see [Le1, GSS]). We obtain in the meantime a single obstruction with two parameters for the dihedral, semidihedral, and quaternion groups of order 16. In Theorem 3.2, given that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, we again find a single obstruction that is valid for the three 2-groups instead of investigating each Brauer problem separetely.

2. EMBEDDING PROBLEMS WITH CYCLIC 2-KERNELS

Let K/k be a Galois extension with Galois group F, and consider the embedding problem given by K/k and the finite group extension

$$1 \to A \to G \underset{\pi}{\to} F \to 1. \tag{2.1}$$

Let the kernel A be Abelian of order n, and let a primitive nth root of unity ζ be in K. Then we can form the character group $\widehat{A} = \{\chi\colon A \to K^*\}$, where χ is a homomorphism of A to the group of roots of unity contained in K. We denote the action of F on A by

$$^{\sigma}a = \bar{\sigma}^{-1}a\bar{\sigma}, \quad \sigma \in F, a \in A,$$

where $\bar{\sigma}$ is a preimage of σ in G. We denote the action of F on K by σx for $x \in K$, and introduce the action of F on $\widehat{A}$ by

$$\chi^{\sigma}(a) = \sigma(\chi(^{\sigma^{-1}}a)), \quad \sigma \in F, a \in A.$$

An embedding problem is called *Brauer* if $\chi^{\sigma} = \chi$ for all $\chi \in \widehat{A}$ and $\sigma \in F$. By [ILF, Theorem 3.2], the compatibility condition is necessary and sufficient for solvability of the Brauer problem with Abelian kernel.

We introduce the following notations: $F_{\chi} = \{\sigma \in F, \chi^{\sigma} = \chi\}$, the subgroup of F which acts on certain character $\chi \in \widehat{A}$ trivially; $B_{\chi} = \operatorname{Ker}\chi$, the kernel of χ; and $A_{\chi} = A/B_{\chi}, H_{\chi} = \pi^{-1}(F_{\chi}), G_{\chi} = H_{\chi}/B_{\chi}$ and K_{χ}, the fixed field of F_{χ}. Then the compatibility condition for the embedding problem $(K/k, G, A)$ holds if and only if the associated problems $(K/K_{\chi}, G_{\chi}, A_{\chi})$ related to the group extensions

$$1 \to A_{\chi} \to G_{\chi} \underset{\pi_{\chi}}{\to} F_{\chi} \to 1 \tag{2.2}$$

are solvable for all characters χ. It is clear that A_{χ} is a cyclic group and χ is a generator of the character group of A_{χ}.

In cohomological terms, the group μ_n of nth roots of unity is embedded in the multiplicative group K^* of the field K. Then the character χ induces a homomorphism $\overline{\chi}$: $H^2(F_{\chi}, A_{\chi}) \to H^2(F_{\chi}, K^*)$, and the compatibility condition can be stated as $\overline{\chi}(c_{\chi}) = 1$ for all $\chi \in \widehat{A}$, where c_{χ} is the 2-coclass of (2.2) in $H^2(F_{\chi}, A_{\chi})$.

In fact, we need not consider all of these problems. It suffices to consider the problems $(K/K_{\chi}, G_{\chi}, A_{\chi})$, where χ runs through a set of representatives of the conjugate classes in $\widehat{A}$, considered as an F-module. In particular, for a Brauer problem with cyclic kernel, all characters are powers of a certain $\chi, F_{\chi} = F$ and $K_{\chi} = k$, so the compatibility condition obtains the form $\overline{\chi}(c) = 1$, where c is the class of (2.1) in $H^2(F, A)$ and $\overline{\chi}$: $H^2(F, A) \to H^2(F, K^*)$.

Now let $A = C_4$ be generated by an element a, and let $i \in K$ be a primitive fourth root of unity. Then $\widehat{A}$ is generated by an element χ such that $\chi(a) = i$. If the embedding problem $(K/k, G, C_4)$ is not Brauer, then there exists $\kappa \in F$ such that $\chi^{\kappa} = \chi^{-1}$, so $N = F_{\chi}$ is a subgroup of F of index 2. Hence N is the maximal subgroup of F which acts on C_4 and μ_4 in the same way. We see that $\operatorname{Ker}\chi = \{1\}, A_{\chi} = C_4, G_{\chi} = H_{\chi} = \pi^{-1}(N), \operatorname{Ker}\chi^2 = \{1, a^2\} \cong C_2, A_{\chi^2} \cong C_2, F_{\chi^2} \cong F$, and $G_{\chi^2} \cong G/C_2$. The conjugate classes in $\widehat{A}$ are $\{1\}, \{\chi, \chi^{-1}\}$, and $\{\chi^2\}$. Therefore, the compatibility condition holds if and only if the associated problems $(K/k_1, \pi^{-1}(N), C_4)$ and $(K/k, G/C_2, C_2)$, related to the group extensions

$$1 \to \mu_4 \to \pi^{-1}(N) \underset{\pi}{\to} N \to 1,$$

where $k_1 = K_\chi$ is the fixed field of N and

$$1 \to C_2 \to G/C_2 \underset{\pi}{\to} F \to 1$$

are solvable. By [ILF, Section 4] the compatibility condition for embedding problems with cyclic kernel of order 4 is also sufficient for solvability (see also [MZ, Corollary 3.3]). We define homomorphisms e, f, and g from F in $\{+1, -1\}$ by ${}^{\sigma}a = a^{e_\sigma}$, $\sigma i = i^{f_\sigma}$, and $g_\sigma = e_\sigma f_\sigma$. Then $N = \{\sigma \in F, g_\sigma = 1\}$, so we obtain from another point of view Ledet's result [Le2, Theorem 1.1].

Now let $A = C_{2^n}, n \geq 2$, be generated by an element a, let K contain a primitive 2^nth root of unity ζ, and let χ: $C_{2^n} \to K^*$ be a generator of $\widehat{C}_{2^n}$. Clearly, for an odd m we have $F_\chi = F_{\chi^m}$ and $\mathrm{Ker}\chi = \mathrm{Ker}\chi^m = \{1\}$. Also let F_χ be of index 2 in F. Then from $F_\chi \subset F_{\chi^{2m}}$, it follows that $F_\chi = F_{\chi^{2m}}$ or $F = F_{\chi^{2m}}$.

If $F_\chi = F_{\chi^{2m}}$, then we obtain the Brauer problem $(K/K_\chi, \pi^{-1}(F_\chi)/B_{2m}, C_{2^n}/B_{2m})$, which is an associated problem of the first kind to the problem $(K/K_\chi, \pi^{-1}(F_\chi), C_{2^n})$. Here, for abuse of notation, $B_{2m} = B_{\chi^{2m}}$.

If $F = F_{\chi^{2m}}$, then we obtain the Brauer problem $(K/k, G/B_{2m}, C_{2^n}/B_{2m})$. From this type of problem, we need to consider only the one with the "biggest" kernel. Namely, let $\chi^\sigma = \chi^{m_\sigma}$, and $m_\sigma \in \mathbb{N}$, and since F_χ is of index 2, we have $m_\sigma \in \{1, l\}$, where l is odd such that $l^2 \equiv 1 \pmod{2^n}$. Hence $(\chi^{2m})^\sigma = \chi^{2m}$ for all $\sigma \in F$ if and only if $2ml \equiv 2m \pmod{2^n}$, i.e., $ml \equiv m \pmod{2^{n-1}}$. Now let m_0 be the minimal natural number such that $1 \leq m_0 \leq 2^{n-2}$ and $m_0 l \equiv m_0 \pmod{2^{n-1}}$. Thus, if m is such that $F = F_{\chi^{2m}}$, then $B_{2m_0} \subset B_{2m}$, so we get the isomorphisms $(C_{2^n}/B_{2m_0})/(B_{2m}/B_{2m_0}) \cong C_{2^n}/B_{2m}$ and $(G/B_{2m_0})/(B_{2m}/B_{2m_0}) \cong G/B_{2m}$. Therefore, the embedding problem $(K/k, G/B_{2m}, C_{2^n}/B_{2m})$ is an associated problem of the first kind to the problem $(K/k, G/B_{2m_0}, C_{2^n}/B_{2m_0})$. In this way the compatibility condition of the problem $(K/k, G, C_{2^n})$ is equivalent to the solvability of the two problems $(K/K_\chi, \pi^{-1}(F_\chi), C_{2^n})$ and $(K/k, G/B_{2m_0}, C_{2^n}/B_{2m_0})$. There are two important cases:

1. If $l \equiv 1 \pmod{2^{n-1}}$, then $m_0 = 1$ and $B_2 \cong C_2$, and so the latter embedding problem is $(K/k, G/C_2, C_{2^{n-1}})$.

2. If $l \equiv -1 \pmod 4$, then $m_0 = 2^{n-2}$ and $B_{2^{n-1}} \cong C_{2^{n-1}}$, and so the latter embedding problem is $(K/k, G/C_{2^{n-1}}, C_2)$.

In this way we once again obtain the following results, which are proved explicitly in [Mi2].

THEOREM 2.1. *Let K/k be a finite Galois extension with Galois group F, and let $\zeta \in K$ be a primitive 2^nth root of unity $(n > 1)$. Consider the group extension*

$$1 \to C_{2^n} \to G \underset{\pi}{\to} F \to 1 \tag{2.3}$$

such that $e_\sigma, f_\sigma \in \{+1, -1\}$ for all $\sigma \in F$. Let k_1 be the fixed field of $N =$ Ker$g$. Then the embedding problem $(K/k, G, C_{2^n})$ is solvable if and only if the embedding problems $(K/k_1, \pi^{-1}(N), \mu_{2^n})$ and $(K/k, G/C_{2^{n-1}}, \mu_2)$ are solvable.

COROLLARY 2.2. *Let K/k be a finite Galois extension with Galois group F, and let ζ be a primitive 2^nth root of unity $(n > 1)$ such that $\zeta + \zeta^{-1} \in k$, $i(\zeta - \zeta^{-1}) \in k$ and $i \notin K$. Let*

$$1 \to C_{2^n} \to G \underset{\pi}{\to} F \to 1$$

be a group extension. Extend the elements $\sigma \in F$ to $K(i)$ by $\sigma i = i$, and let κ be the generator of Gal$(K(i)/K)$. *Let $k(\sqrt{b})$ be the fixed field of $N =$ Ker$g$ and $k_1 = k(i\sqrt{b})$. Then* Gal$(K(i)/k_1) \cong F$, *and the embedding problem $(K/k, G, C_{2^n})$ is solvable if and only if the embedding problems $(K(i)/k_1, G, \mu_{2^n})$ and $(K/k, G/C_{2^{n-1}}, \mu_2)$ are solvable.*

The foregoing results imply that the embedding problem $(K/k, G, C_{2^n})$ has two obstructions corresponding to the two reduced Brauer problems. In the following section we decompose each obstruction into a product of quaternion classes.

For $a, b \in k^*$, we denote by (a, b) the equivalence class in Br(k) of the quaternion algebra generated over k by elements i and j with relations $i^2 = a$, $j^2 = b$ and $ij = -ji$. We note that when elements i and $j \neq 0$ with relations $i^2 = a^2$, $j^2 = 0$, and $ij = -ji$ show up in a centraliser (see Theorem 3.2 below), they demonstrate that the centraliser is split, even though they do not generate it.

3. THE DIHEDRAL, SEMIDIHEDRAL, AND QUATERNION GROUPS

In this section we investigate embedding problems involving the dihedral (D_{2^n}), semidihedral (SD_{2^n}), and quaternion (Q_{2^n}) groups of order 2^n, $n \geq 4$. Their presentations are as follows:

$$D_{2^n} \cong \langle \sigma, \tau | \sigma^{2^{n-1}} = \tau^2 = 1, \tau\sigma = \sigma^{-1}\tau \rangle$$

$$SD_{2^n} \cong \langle \sigma, \tau | \sigma^{2^{n-1}} = \tau^2 = 1, \tau\sigma = \sigma^{2^{n-2}-1}\tau \rangle$$

$$Q_{2^n} \cong \langle \sigma, \tau | \sigma^{2^{n-1}} = 1, \tau^2 = \sigma^{2^{n-2}}, \tau\sigma = \sigma^{-1}\tau \rangle$$

First, consider the case where $n = 4$. Let $K/k = k(\sqrt{r(\alpha+\beta\sqrt{a})}, \sqrt{b})/k$ be a D_8 extension, where a and b are quadratically independent, $r \in k^*$, and $\alpha, \beta \in k$, such that $\alpha^2 - a\beta^2 = ab$. Denote $\varphi = \sqrt{r(\alpha+\beta\sqrt{a})}$ and $\psi = \sqrt{r(\alpha-\beta\sqrt{a})}$. Then $\varphi\psi = r\sqrt{ab}$, and D_8 is generated by elements σ and τ such that

$$\sigma\colon \varphi \mapsto \psi, \psi \mapsto -\varphi, \sqrt{b} \mapsto \sqrt{b}$$
$$\tau\colon \varphi \mapsto \varphi, \psi \mapsto -\psi, \sqrt{b} \mapsto -\sqrt{b}.$$

Now consider the group extension

$$1 \to C_2 = \{\pm 1\} \to G \underset{\substack{s\mapsto\sigma \\ t\mapsto\tau}}{\to} D_8 \to 1,$$

where s and t are preimages in G of σ and τ, respectively, such that $s^4 = -1$, $t^2 = \varepsilon_1$, and $ts = \varepsilon_2 s^3 t$ for $\varepsilon_1 = (-1)^{m_1}$, $\varepsilon_2 = (-1)^{m_2}$, and $m_1, m_2 \in \{0, 1\}$.

The crossed product algebra $\Gamma = (K, D_8, -1)$, corresponding to the extension, contains the following three quaternion subalgebras:

$$Q_1 : i_1 = t, \qquad j_1 = \sqrt{b}$$
$$Q_2 : i_2 = (s+s^3)\sqrt{b}^{m_2}, \qquad j_2 = \sqrt{a}$$
$$Q_3 : i_3 = s^2\sqrt{b}, \qquad j_3 = (\varphi + \psi s)\sqrt{a}.$$

We see that $i_1^2 = (-1)^{m_1}$, $j_1^2 = b$, $i_2^2 = -2b^{m_2}$, $j_2^2 = a$, $i_3^2 = -b$, and $j_3^2 = 2r\alpha a$. Since Q_1, Q_2, and Q_3 centralize each other, we get

$$[\Gamma] = [Q_1][Q_2][Q_3] = ((-1)^{m_1}, b)(-2b^{m_2}, a)(-b, 2r\alpha a) \in \mathrm{Br}(k).$$

Thus we get the following theorem.

THEOREM 3.1. *The obstructions to the embedding problem* $(K/k, G, C_2)$ *are as folows*:

1. $m_1 = 0, m_2 = 1$ $(G = D_{16}) : (a, 2)(-b, 2r\alpha) \in \mathrm{Br}(k)$
2. $m_1 = m_2 = 1$ $(G = Q_{16}) : (a, 2)(b, b)(-b, 2r\alpha) \in \mathrm{Br}(k)$
3. $m_1 = m_2 = 0$ $(G = SD_{16}) : (a, -2)(-b, 2r\alpha) \in \mathrm{Br}(k)$
4. $m_1 = 1, m_2 = 0$ $(G = SD_{16}) : (a, -2)(b, b)(-b, 2r\alpha) \in \mathrm{Br}(k)$.

Note that we have two distinct obstructions for SD_{16}, since the two corresponding group extensions are nonequivalent. A thorough discussion of the obstructions for the groups of order 16 can be found in [GSS, Ki, Le1].

Now, let K/k be a D_8 extension and let $\zeta \in K$ be a primitive 2^nth root of unity such that $\zeta \notin k$, $\zeta + \zeta^{-1} \in k$, and $i(\zeta - \zeta^{-1}) \in k$.

Then $K/k = k(\sqrt[4]{a}, i)$ for some $a \in k\backslash k^2$, and D_8 is generated by elements σ and τ, given by

$$\sigma\colon \sqrt[4]{a} \mapsto i\sqrt[4]{a}, i \mapsto i;\ \tau\colon \sqrt[4]{a} \mapsto \sqrt[4]{a}, i \mapsto -i$$

(in particular, $\sigma(\zeta) = \zeta$ and $\tau(\zeta) = \zeta^{-1}$).

We now turn our attention to the case when G is a group generated by elements s and t, such that s is of order 2^{n+2}, $t^2 = \varepsilon_1$ and $ts = \varepsilon_2 s^{-1} t$, where $\varepsilon_1^2 = \varepsilon_2^2 = 1$. Since $ts^4 = s^{-4}t$, we can put $s^4 = \zeta$, and get the group extension

$$1 \to \mu_{2^n} \underset{\zeta \mapsto s^4}{\to} G \underset{\substack{s \mapsto \sigma \\ t \mapsto \tau}}{\to} D_8 \to 1, \tag{3.1}$$

where we identify the cyclic group $\langle s^4 \rangle$ with the group of 2^nth roots of unity μ_{2^n}. Therefore, we have $s^4 = \zeta$, $t^2 = \varepsilon_1$, and $ts = \varepsilon_2 \zeta^{-1} s^3 t$, where $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$. The group G has an element of order 2^{n+2}, and hence G is isomorphic either to the dihedral, semidihedral, or quaternion group of order 2^{n+3}. Our main result of this section is calculation of the obstruction to the embedding problem $(K/k, G, \mu_{2^n})$ in the following theorem.

THEOREM 3.2. *For the solvability of the embedding problem $(K/k, G, \mu_{2^n})$ for $n \geq 1$, it is necessary that there exists $\alpha_1 \in k^*$ and $\beta_1 \in k$, such that $\alpha_1^2 + a\beta_1^2 = 2 - \zeta - \zeta^{-1}$. In that case the obstruction is*

$$(-1, \varepsilon_1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)\left(a, \varepsilon_2\alpha_1\left(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i}\right)\right) \in \mathrm{Br}(k).$$

Proof. We proceed by induction. For $n = 1$, we have $\zeta = -1$ and let $\alpha_1 = 2$, $\beta_1 = 0 : {\alpha_1}^2 + a{\beta_1}^2 = 2 - \zeta - \zeta^{-1} = 4$. Then we get the obstruction $(-1, \varepsilon_1)(a, 2\varepsilon_2) \in \mathrm{Br}(k)$, which can also be obtained from Theorem 3.1 for $b = -1$. Now let the embedding problems for $n - 1$ be solvable. In particular, the associated problem $(K/k, D_{2^{n+2}}, \mu_{2^{n-1}})$ is solvable (here $\varepsilon_1 = \varepsilon_2 = 1$). Then ζ^2 is a primitive 2^{n-1}th root of unity and $2 - \zeta^2 - \zeta^{-2} = \left(\frac{\zeta - \zeta^{-1}}{i}\right)^2$, so we can let $\alpha_1 = \frac{\zeta - \zeta^{-1}}{i}$ and $\beta_1 = 0$. Thus by the induction assumption, the obstruction to $(K/k, D_{2^{n+2}}, \mu_{2^{n-1}})$ is

$$\begin{aligned}
&((\zeta + \zeta^{-1})^2, 0)\left(a, \frac{\zeta - \zeta^{-1}}{i}\left(2\frac{\zeta - \zeta^{-1}}{i} - \frac{\zeta^2 - \zeta^{-2}}{i}\right)\right) \\
&= \left(a, \frac{\zeta - \zeta^{-1}}{i}\left(2\frac{\zeta - \zeta^{-1}}{i} - \frac{\zeta - \zeta^{-1}}{i}(\zeta + \zeta^{-1})\right)\right) \\
&= \left(a, \left(\frac{\zeta - \zeta^{-1}}{i}\right)^2 (2 - \zeta - \zeta^{-1})\right) = (a, 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k).
\end{aligned}$$

Further,

$$(2-\zeta-\zeta^{-1})(2+\zeta+\zeta^{-1}) = 4-(\zeta+\zeta^{-1})^2$$
$$= 2-\zeta^2-\zeta^{-2} = \left(\frac{\zeta-\zeta^{-1}}{i}\right)^2 \in k^2$$

and

$$\left(1+\frac{\zeta+\zeta^{-1}}{2}\right)^2 + \left(\frac{\zeta-\zeta^{-1}}{2i}\right)^2$$
$$= 1+\zeta+\zeta^{-1}+\frac{\zeta^2+\zeta^{-2}}{4}+\frac{1}{2}-\frac{\zeta^2+\zeta^{-2}}{4}+\frac{1}{2}$$
$$= 2+\zeta+\zeta^{-1}.$$

Hence both $2+\zeta+\zeta^{-1}$ and $2-\zeta-\zeta^{-1}$ are sums of two squares in k. Thus we obtain that $(-a, 2-\zeta-\zeta^{-1}) = 1 \in \mathrm{Br}(k)$ (or, equivalently, $(-a, 2+\zeta+\zeta^{-1}) = 1 \in \mathrm{Br}(k)$) is necessary for solvability of the embedding problem $(K/k, G, \mu_{2^n})$ for $n > 1$.

Now let $\alpha_2 \in k^*$ and $\beta_2 \in k$ be such that $\alpha_2^2 + a\beta_2^2 = 2+\zeta+\zeta^{-1}$. The connection between α_2, β_2 and α_1, β_1 is given by

$$\alpha_1^2 + a\beta_1^2 = 2-\zeta-\zeta^{-1} = \frac{2-\zeta^2-\zeta^{-2}}{2+\zeta+\zeta^{-1}}$$
$$= (2+\zeta+\zeta^{-1})\left(\frac{\zeta-\zeta^{-1}}{i(2+\zeta+\zeta^{-1})}\right)^2.$$

We let $\gamma = \frac{\zeta-\zeta^{-1}}{i(2+\zeta+\zeta^{-1})} \in k$, $\alpha_2 = \frac{\alpha_1}{\gamma}$, and $\beta_2 = \frac{\beta_1}{\gamma}$ and get $\alpha_2^2 + a\beta_2^2 = 2+\zeta+\zeta^{-1}$.

Let Γ be the algebra representing the obstruction. Then Γ is generated by two elements u and v over K such that $u^4 = \zeta$, $v^2 = \varepsilon_1$, $vu = \varepsilon_2 u^{-1} v = \varepsilon_2 \zeta^{-1} u^3 v$, $ux = \sigma(x)u$, and $vx = \tau(x)v$ for $x \in K$. Then Γ contains the following three quaternion subalgebras:

$$Q_1 : i_1 = i, \qquad j_1 = v$$
$$Q_2 : i_2 = (1+\zeta^{-1})u^2, \qquad j_2 = \sqrt[4]{a}(\alpha_2 + \beta_2\sqrt{a} + \varepsilon_2(1+\zeta^{-1})u^2)$$
$$Q_3 : i_3 = \sqrt{a}, \qquad j_3 = [-(1+i)(1+\zeta^{-1}) + \alpha_2(1+i) + (1-i)\beta_2\sqrt{a}]u + \varepsilon_2\zeta^{-1}[-(1-i)(1+\zeta) + \alpha_2(1-i) + (1+i)\beta_2\sqrt{a}]u^3.$$

Calculations show that $i_1^2 = -1$, $j_1^2 = \varepsilon_1$, $i_2^2 = 2+\zeta+\zeta^{-1}$, $j_2^2 = 2\alpha_2\beta_2 a$, $i_3^2 = a$, and $j_3^2 = \varepsilon_2 4\alpha_2(2\alpha_2 - 2 - \zeta - \zeta^{-1})$. Also, $i_s j_s = -j_s i_s$, $1 \le s \le 3$, and the generators of each algebra pairwise commute with the generators of the other two. We are forced to omit the monstrous verification, however.

Thus finally we obtain

$$\begin{aligned}[\Gamma] &= [Q_1][Q_2][Q_3] \\ &= (-1, \varepsilon_1)(2+\zeta+\zeta^{-1}, \alpha_2\beta_2)(a, \varepsilon_2\alpha_2(2\alpha_2 - 2 - \zeta - \zeta^{-1})) \\ &= (-1, \varepsilon_1)(2+\zeta+\zeta^{-1}, \alpha_1\beta_1)\left(a, \varepsilon_2\frac{\alpha_1}{\gamma}\left(2\frac{\alpha_1}{\gamma} - 2 - \zeta - \zeta^{-1}\right)\right) \\ &= (-1, \varepsilon_1)(2+\zeta+\zeta^{-1}, \alpha_1\beta_1)\left(a, \varepsilon_2\alpha_1\left(2\alpha_1 - \frac{\zeta-\zeta^{-1}}{i}\right)\right) \in \mathrm{Br}(k). \quad \blacksquare\end{aligned}$$

Remark 3.1. If it happens that for $n \geq 3$ we have $\alpha_1 = 0$ and $a\beta_1^2 = 2 - \zeta - \zeta^{-1}$, then we can put $\alpha_1' = \frac{2}{3+\zeta+\zeta^{-1}}(\frac{\zeta-\zeta^{-1}}{i})$ and $\beta_1' = \frac{1+\zeta+\zeta^{-1}}{3+\zeta+\zeta^{-1}}\beta_1$, and hence ${\alpha_1'}^2 + a{\beta_1'}^2 = 2 - \zeta - \zeta^{-1}$. For $n = 2$, this works well if k has characteristic $\neq 3$, then we simply have $\alpha_1' = \frac{4}{3}$ and $\beta_1' = \frac{1}{3}\beta_1$. If k has characteristic 3, then we can put $\alpha_1' = a - 1/a$ and $\beta_1' = (a + 1/a)\beta_1$, so ${\alpha_1'}^2 + a{\beta_1'}^2 = 2$.

Remark 3.2. For $n \geq 3$, we have that $\zeta^{2^s} + \zeta^{-2^s} \in k$ and $2 \in k^2$. From the proof we also get that $(a, 2 - \zeta^{2^s} - \zeta^{-2^s}) = 1 \in \mathrm{Br}(k),\ 0 \leq s \leq n-1$, is necessary for solvability of the embedding problem.

Helping our consideration is the following lemma, obtained in [Mi2].

LEMMA 3.3. *Let $\zeta \in k$ be a primitive 2^nth root of unity* ($n \geq 1$), *and let $i \in k$. For the embedding problem given by a C_4 extension $k(\sqrt[4]{a})/k$ and the group extension*

$$1 \to \mu_{2^n} \hookrightarrow C_{2^{n+2}} \to C_4 \to 1 \tag{3.2}$$

to be solvable, it is necessary that there exist $\alpha', \beta' \in k, \alpha' \neq 0$, such that ${\alpha'}^2 - a{\beta'}^2 = \zeta$. In that case, the obstruction is $(a, \alpha')(\zeta, \alpha'\beta') \in \mathrm{Br}(k)$.

Now consider the group extension

$$1 \to C_{2^n} \to G \underset{\substack{x\mapsto\sigma \\ y\mapsto\tau}}{\to} D_8 \to 1, \tag{3.3}$$

where G is generated by elements x of order 2^{n+2} and y. We then have four non-equivalent group extensions lifting an element of order 4 to one of order 2^{n+2}:

$$1 \to C_{2^n} \to D_{2^{n+3}} \underset{\substack{x\mapsto\sigma \\ y\mapsto\tau}}{\to} D_8 \to 1 \tag{3.4a}$$

$$1 \to C_{2^n} \to Q_{2^{n+3}} \underset{\substack{x\mapsto\sigma \\ y\mapsto\tau}}{\to} D_8 \to 1 \tag{3.4b}$$

(3.4c) $$1 \to C_{2^n} \to SD_{2^{n+3}} \underset{\substack{x\mapsto\sigma \\ y\mapsto\tau}}{\to} D_8 \to 1$$

(3.4d) $$1 \to C_{2^n} \to SD_{2^{n+3}} \underset{\substack{x\mapsto\sigma \\ yx\mapsto\tau}}{\to} D_8 \to 1.$$

Assume again that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, so the location of ζ in K/k is determined by the location of i. Recall that $K/k = k(\sqrt{r(\alpha + \beta\sqrt{a})}, \sqrt{b})/k$, where $r \in k^*$ and $\alpha, \beta \in k$, such that $\alpha^2 - a\beta^2 = ab$.

We find the obstructions to the embedding problems related to group extensions (3.4a)–(3.4d) in all five possible cases.

1. $i \in k$. Then $\zeta \in k$, so $\sigma\zeta = \tau\zeta = \zeta$, $\chi^\sigma = \chi$, and $\chi^\tau = \chi^{-1}$. Hence $F_\chi = \langle\sigma\rangle$ and $K_\chi = k(\sqrt{b})$. By Theorem 2.1, the embedding problems related to group extensions (3.4a)–(3.4d) are solvable $\Leftrightarrow$ the embedding problem given by $K/k(\sqrt{b})$ and (3.2), respectively, by K/k and

$$1 \to C_2 \hookrightarrow D_{16} \cong G/C_{2^{n-1}} \to D_8 \to 1$$

are solvable. Here $K/k(\sqrt{b}) = k(\sqrt[4]{a'})/k(\sqrt{b})$ for $a' = [2r(\beta - i\sqrt{b})]^2 a$. By Theorem 3.1 and Lemma 3.3, the obstructions for each embedding problem are $(ab, 2)(-b, r\alpha) \in \mathrm{Br}(k)$ and $(a, \alpha')(\zeta, \alpha'\beta') \in \mathrm{Br}(k(\sqrt{b}))$, where we must have $\alpha' \in k(\sqrt{b})^*$, and $\beta' \in k(\sqrt{b})$ such that $\alpha'^2 - a\beta'^2 = \zeta$.

2. $a = -1$. Then $\sigma\zeta = \zeta^{-1}, \tau\zeta = \zeta$, and $\chi^\sigma = \chi^\tau = \chi^{-1}$. Hence $F_\chi = \langle\sigma^2, \tau\sigma\rangle \cong C_2 \times C_2$ and $K_\chi = k(i\sqrt{b})$. The embedding problem $(K/k, G, C_{2^n})$ is solvable $\Leftrightarrow$ the embedding problems $(K/k(i\sqrt{b}), \pi^{-1}C_2^2, \mu_{2^n})$ and $(K/k, D_{16}, \mu_2)$ are solvable. Then we must have $(-b, 2\alpha r) = 1 \in \mathrm{Br}(k)$, and the obstructions for each embedding problem are obtained as follows:

(3.4a), $\pi^{-1}C_2^2 \cong D_{2^{n+2}}$: The embedding problem $(k(\sqrt{a'}, i)/k(i\sqrt{b}), D_{2^{n+2}}, \mu_{2^n})$ for $a' = (\varphi + \psi)^2 = 2r(\alpha + i\sqrt{b})$ is solvable $\Leftrightarrow$ the embedding problem $(k(\sqrt[4]{a''}, i)/k(i\sqrt{b}), D_{2^{n+2}}, \mu_{2^{n-1}})$ is solvable for some $a'' = r'^2 a'$, and $r' \in k(i\sqrt{b})$. Thus the obstruction to the embedding problem $(K/k, D_{2^{n+3}}, C_{2^n})$ is $(a'', 2 - \zeta - \zeta^{-1}) = (2r(\alpha + i\sqrt{b}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(i\sqrt{b}))$.

(3.4b), $\pi^{-1}C_2^2 \cong Q_{2^{n+2}}$: The embedding problem $(k(\sqrt{a'}, i)/k(i\sqrt{b}), Q_{2^{n+2}}, \mu_{2^n})$ for $a' = 2r(\alpha + i\sqrt{b})$ is solvable $\Leftrightarrow$ the embedding problem $(k(\sqrt[4]{a''}, i)/k(i\sqrt{b}), Q_{2^{n+2}}, \mu_{2^{n-1}})$ is solvable for some $a'' = r'^2 a'$, and $r' \in k(i\sqrt{b})$. Thus the obstruction to the embedding problem $(K/k, Q_{2^{n+3}}, C_{2^n})$ is $(-1, -1)(2r(\alpha + i\sqrt{b}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(i\sqrt{b}))$.

(3.4c), $\pi^{-1}C_2^2 \cong Q_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, SD_{2^{n+3}}, C_{2^n})$ is $(-1, -1)(2r(\alpha + i\sqrt{b}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(i\sqrt{b}))$.

(3.4d), $\pi^{-1}C_2^2 \cong D_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, SD_{2^{n+3}}, C_{2^n})$ is $(2r(\alpha + i\sqrt{b}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(i\sqrt{b}))$.

3. $b = -1$. This is the case considered in Theorem 3.2. We may assume that $r = \beta = 1$ and $\alpha = 0$. We must have $\alpha_1 \in k^*$, and $\beta_1 \in k$ such that $\alpha_1^2 + a\beta_1^2 = 2 - \zeta - \zeta^{-1}$. Then the obstructions are

(3.4a): $(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k)$
(3.4b): $(-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k)$.
(3.4c): $(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k)$.
(3.4d): $(-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k)$.

4. $ab = -1$. Then $\sigma\zeta = \zeta^{-1}, \tau\zeta = \zeta^{-1}$, $\chi^\sigma = \chi^{-1}$, and $\chi^\tau = \chi$. Hence $F_\chi = \langle \sigma^2, \tau \rangle \cong C_2 \times C_2$ and $K_\chi = k(\sqrt{a})$. The embedding problem $(K/k, G, C_{2^n})$ is solvable $\Leftrightarrow$ the embedding problems $(K/k(\sqrt{a}), \pi^{-1}C_2^2, \mu_{2^n})$ and $(K/k, D_{16}, \mu_2)$ are solvable. Then we must have $(-b, \alpha r) = 1 \in \mathrm{Br}(k)$, and the obstructions for each embedding problem are obtained as before:

(3.4a), $\pi^{-1}C_2^2 \cong D_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, D_{2^{n+3}}, C_{2^n})$ is $(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(\sqrt{a}))$.
(3.4b), $\pi^{-1}C_2^2 \cong Q_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, Q_{2^{n+3}}, C_{2^n})$ is $(-1, -1)(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}((k(\sqrt{a}))$.
(3.4c), $\pi^{-1}C_2^2 \cong D_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, D_{2^{n+3}}, C_{2^n})$ is $(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(\sqrt{a}))$.
(3.4d), $\pi^{-1}C_2^2 \cong Q_{2^{n+2}}$: The obstruction to the embedding problem $(K/k, D_{2^{n+3}}, C_{2^n})$ is $(-1, -1)(r(\alpha + \beta\sqrt{a}), 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k(\sqrt{a}))$.

5. a, b and -1 are quadratically independent. Let κ generate $\mathrm{Gal}(K(i)/K)$, and identify $\mathrm{Gal}(K/k)$ with $\mathrm{Gal}(K(i)/k(i))$. Then the embedding problem $(K/k, G, C_{2^n})$ is solvable $\Leftrightarrow$ the embedding problem given by $K(i)/k(i)$ and

$$1 \to C_{2^n} \to G \times C_2 \to D_8 \times C_2 \to 1$$

is solvable. Here $(D_8 \times C_2)_\chi = \langle \sigma, \tau\kappa \rangle \cong D_8$ and $K(i)_\chi = k(i\sqrt{b})$. The restricted embedding problem is then given by $K(i)/k(i\sqrt{b})$ and

$$1 \to \mu_{2^n} \to G \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau\kappa}}{\to} D_8 \to 1.$$

We must have $(ab, 2)(-b, \alpha r) = 1 \in \mathrm{Br}(k)$, $\alpha_1 \in k(i\sqrt{b})^*$, and $\beta_1 \in k(i\sqrt{b})$, such that $\alpha_1^2 + a'\beta_1^2 = 2 - \zeta - \zeta^{-1}$ and $K(i) = k(i\sqrt{b})(\sqrt[4]{a'}, i)$, where $a' = [2r(\alpha + i\sqrt{b})]^2 a$. Then the obstructions are:

(3.4a): $(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k(i\sqrt{b}))$.
(3.4b): $(-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, \alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k(i\sqrt{b}))$.
(3.4c): $(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k(i\sqrt{b}))$.
(3.4d): $(-1, -1)(2 + \zeta + \zeta^{-1}, \alpha_1\beta_1)(a, -\alpha_1(2\alpha_1 - \frac{\zeta - \zeta^{-1}}{i})) \in \mathrm{Br}(k(i\sqrt{b}))$.

Now let $k(\sqrt{a}, \sqrt{b})/k$ be a C_2^2 extension generated by elements ρ_1 and ρ_2 such that

$$\rho_1\colon \sqrt{a} \mapsto -\sqrt{a}, \sqrt{b} \mapsto \sqrt{b};\ \rho_2\colon \sqrt{a} \mapsto \sqrt{a}, \sqrt{b} \mapsto -\sqrt{b}.$$

Consider the embedding problem given by $k(\sqrt{a}, \sqrt{b})/k$ and

$$1 \to C_{2^{n+1}} \to G \underset{\substack{x \mapsto \rho_1 \\ y \mapsto \rho_2}}{\to} C_2^2 \to 1, \tag{3.5}$$

where the group G is generated by elements x and y such that x is of order 2^{n+2}, $y^2 = x^{2^{n+1}}$ or $y^2 = 1$, and $yx = x^{-1}y$ or $yx = x^{2^{n+1}-1}y$. Hence G is isomorphic to either $D_{2^{n+3}}$, $Q_{2^{n+3}}$, or $SD_{2^{n+3}}$ groups. Obviously, this embedding problem is solvable if and only if $k(\sqrt{a}, \sqrt{b})/k$ can be embedded in a D_8 extension K/k and the embedding problem $(K/k, G, C_{2^n})$ is solvable.

Again, extension (3.5) generates four group extensions:

$$1 \to C_{2^{n+1}} \to D_{2^{n+3}} \underset{\substack{x \mapsto \rho_1 \\ y \mapsto \rho_2}}{\to} C_2^2 \to 1 \tag{3.6a}$$

$$1 \to C_{2^{n+1}} \to Q_{2^{n+3}} \underset{\substack{x \mapsto \rho_1 \\ y \mapsto \rho_2}}{\to} C_2^2 \to 1 \tag{3.6b}$$

$$1 \to C_{2^{n+1}} \to SD_{2^{n+3}} \underset{\substack{x \mapsto \rho_1 \\ y \mapsto \rho_2}}{\to} C_2^2 \to 1 \tag{3.6c}$$

$$1 \to C_{2^{n+1}} \to SD_{2^{n+3}} \underset{\substack{x \mapsto \rho_1 \\ yx \mapsto \rho_2}}{\to} C_2^2 \to 1. \tag{3.6d}$$

We write down the obstructions to the Brauer problems for $b = -1$, related to extensions (3.6a)–(3.6d).

Let ζ be a primitive 2^{n+1}th root of unity ($n > 1$) such that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$. We can let $\alpha_1 = \frac{\zeta - \zeta^{-1}}{i}$, $\beta_1 = 0 : \alpha_1^2 + a\beta_1^2 = (\frac{\zeta - \zeta^{-1}}{i})^2 = 2 - \zeta^2 - \zeta^{-2}$. Then the obstructions are

(3.6a): $(a, 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k)$
(3.6b): $(-1, -1)(a, 2 - \zeta - \zeta^{-1}) \in \mathrm{Br}(k)$
(3.6c): $(a, -2 + \zeta + \zeta^{-1}) \in \mathrm{Br}(k)$
(3.6d): $(-1, -1)(a, -2 + \zeta + \zeta^{-1}) \in \mathrm{Br}(k)$.

We proceed by investigating the embedding problem given by $k(\sqrt{b})/k$ and

$$1 \to C_{2^{n+2}} \to G \to C_2 \to 1, \tag{3.7}$$

where the group G again is isomorphic to either the $D_{2^{n+3}}$, the $Q_{2^{n+3}}$, or the $SD_{2^{n+3}}$ group. Obviously, this embedding problem is solvable if and only if there exists $a \in k$ such that a and b are quadratically independent and the embedding problem given by $k(\sqrt{a}, \sqrt{b})/k$ and (3.5) is solvable.

Let ζ be a primitive 2^{n+2}th root of unity ($n > 1$), such that $\zeta + \zeta^{-1} \in k$ and $i(\zeta - \zeta^{-1}) \in k$, and let $|k/k^2| \geq 4$. Again, we write down the obstructions to the Brauer problems for $b = -1$:

(3.6a): We have $(a, 2 - \zeta^2 - \zeta^{-2}) = 1 \in \mathrm{Br}(k)$ for all $a \in k$ such that a and -1 are quadratically independent. Therefore, there is no obstruction, and it is easy to see that $k(\sqrt[2^{n+2}]{a}, i)/k$ is a solution to the embedding problem $(k(i)/k, D_{2^{n+3}}, \mu_{2^{n+2}})$.

(3.6b): The obstruction is $(-1, -1) \in \mathrm{Br}(k)$. This is exactly the same result obtained in [MZ, Example 3.4].

(3.6c): $(a, -1) \in \mathrm{Br}(k)$

(3.6d): $(-a, -1) \in \mathrm{Br}(k)$.

Thus the embedding problem $(k(i)/k, D_{2^{n+3}}, \mu_{2^{n+2}})$ is solvable $\Leftrightarrow$ $|k/k^2| \geq 4$; $(k(i)/k, Q_{2^{n+3}}, \mu_{2^{n+2}})$ is solvable $\Leftrightarrow |k/k^2| \geq 4$ and $(-1, -1) \in \mathrm{Br}(k)$; and $(k(i)/k,\ SD_{2^{n+3}}, \mu_{2^{n+2}})$ is solvable in both cases $\Leftrightarrow |k/k^2| \geq 4$ and k is not quadratically closed.

Note that all of the obstructions in this section hold for "proper" solutions (i.e., Galois extensions), since we have $\mathrm{rank}(G) = \mathrm{rank}(C_2^2) = \mathrm{rank}(D_8) = 2$.

Finally, for $\zeta = i$ we can consider the group extension

(3.8) $$1 \to C_4 \to G \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\to} D_8 \to 1,$$

where G is isomorphic to either the D_{32}, SD_{32}, or Q_{32} group. Then the obstruction to the Brauer problem is

$$(-1, \varepsilon_1)(2, \alpha_1\beta_1)(a, \varepsilon_2\alpha_1(\alpha_1 - 1)) \in \mathrm{Br}(k),$$

where $\alpha_1 \in k^*$, and $\beta_1 \in k$, such that $\alpha_1^2 + a\beta_1^2 = 2$. This coincides with Ledet's result in [Le2].

Again, the group extension (3.8) generates four extensions:

(3.9a) $$1 \to C_4 \to D_{32} \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\to} D_8 \to 1$$

(3.9b) $$1 \to C_4 \to Q_{32} \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\to} D_8 \to 1$$

(3.9c) $$1 \to C_4 \to SD_{32} \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\to} D_8 \to 1$$

(3.9d) $$1 \to C_4 \to SD_{32} \underset{\substack{x \mapsto \sigma \\ y \mapsto \tau}}{\to} D_8 \to 1.$$

We conclude the paper with several examples on Brauer problems related to extensions (3.9a)–(3.9d) over the rational field.

EXAMPLE 3.1. Consider the embedding problem $(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}, G, \mu_4)$. We put $\alpha_1 = \frac{4}{3}, \beta_1 = \frac{1}{3} : \alpha_1^2 + 2\beta_1^2 = 2$, so the obstruction is $(-1, \varepsilon_1)(2, \frac{4}{9})$

$(2, \varepsilon_2 \frac{4}{9}) = (-1, \varepsilon_1) \in \mathrm{Br}(\mathbb{Q})$. Therefore, the embedding problems given by (3.9a) and (3.9c) are solvable, but those given by (3.9b) and (3.9d) are not.

EXAMPLE 3.2. Consider the embedding problem $(\mathbb{Q}(\sqrt[4]{7}, i)/\mathbb{Q}, G, \mu_4)$. We put $\alpha_1 = \beta_1 = \frac{1}{2} : \alpha_1^2 + 7\beta_1^2 = 2$, so the obstruction is $(-1, \varepsilon_1)(2, \frac{1}{4})$ $(7, -\varepsilon_2 \frac{1}{4}) = (-1, \varepsilon_1)(7, -\varepsilon_2)$. The obstructions for each embedding problem are

(3.9a): $(7, -1) \neq 1 \in \mathrm{Br}(\mathbb{Q})$
(3.9b): $(-7, -1) \neq 1 \in \mathrm{Br}(\mathbb{Q})$
(3.9c): $(-1, 1)(7, 1) = 1 \in \mathrm{Br}(\mathbb{Q})$
(3.9d): $(-1, -1) \neq 1 \in \mathrm{Br}(\mathbb{Q})$.

Therefore, the embedding problems given by (3.9a), (3.9b), and (3.9d) are not solvable, but the embedding problem given by (3.9c) is solvable.

Of course, for an arbitrary rational number a, it is very hard to determine whether the product of these three quaternion algebras is split in $\mathrm{Br}(\mathbb{Q})$. Computer-assisted calculations give the following example, where the embedding problem given by (3.9a) is solvable but the other embedding problems are not.

EXAMPLE 3.3. Consider the embedding problem $(\mathbb{Q}(\sqrt[4]{-27887}, i)/\mathbb{Q}, G, \mu_4)$. We put $\alpha_1 = 167, \beta_1 = 1 : \alpha_1^2 - 27887\beta_1^2 = 2$. Using the technique developed in [Mi1], we can link the splitting of a quaternion algebra in $\mathrm{Br}(\mathbb{Q})$ to Legendre symbols. Since $(\frac{2}{167}) = 1$, we get $(\alpha_1\beta_1, 2) = (167, 2) = 1 \in \mathrm{Br}(\mathbb{Q})$. We have $-27887 = -79 \cdot 353$ and $\alpha_1(\alpha_1 - 1) = 2 \cdot 83 \cdot 167$, so the obstruction is $(-1, \varepsilon_1)(-79 \cdot 353, \varepsilon_2 2 \cdot 83 \cdot 167) \in \mathrm{Br}(\mathbb{Q})$. Note that $167 \equiv 7\ (8), 79 \equiv 7\ (8)$, and $353 \equiv 1\ (8)$. Now $(\frac{2}{79}) = (\frac{2}{353}) = 1$, hence $(-79 \cdot 353, 2) = 1; (\frac{83}{79}) = (\frac{167}{79}) = 1$ and $(\frac{-79}{83}) = (\frac{-79}{167}) = 1$, hence $(-79, 83 \cdot 167) = 1$. Finally, $(\frac{167}{353}) = (\frac{353}{167}) = 1$, hence $(353, 167) = 1$, and $(\frac{83}{353}) = (\frac{353}{83}) = 1$, hence $(353, 83) = 1$.

Thus, if $\varepsilon_2 = 1$, then the obstruction is $(-1, \varepsilon_1)(-79 \cdot 353, 2 \cdot 83 \cdot 167) = (-1, \varepsilon_1)(353,\ 83 \cdot 167) = (-1, \varepsilon_1)(353, 83) = (-1, \varepsilon_1) = 1 \in \mathrm{Br}(\mathbb{Q}) \Leftrightarrow \varepsilon_1 = 1$. If $\varepsilon_2 = -1$ and we assume that $(-1, \varepsilon_1)(-79 \cdot 353, -2 \cdot 83 \cdot 167) = 1 \in \mathrm{Br}(\mathbb{Q})$, then in particular $79 \cdot 353$ is a sum of three integer squares, which is an impossibility since $79 \cdot 353 \equiv 7(8)$. Therefore, we obtain that the embedding problem $(\mathbb{Q}(\sqrt[4]{-27887}, i)/\mathbb{Q}, D_{32}, \mu_4)$ is solvable, but the other embedding problems are not.

REFERENCES

[GSS] H. G. Grundman, T. L. Smith, and J. R. Swallow, Groups of order 16 as Galois groups, *Expo. Math.* **13** (1995), 289–319.

[ILF] V. V. Ishanov, B. B. Lur'e, and D. K. Faddeev, "The Embedding Problem in Galois Theory," American Mathematical Society, Providence, RI, 1997.

[Ki] I. Kiming, Explicit classifications of some 2-extensions of a field of characteristic different from 2, *Cand. J. Math.* **42** (1990), 825–855.
[La] T. Y. Lam, "The Algebraic Theory of Quadratic Forms," Benjamin Cummnigs, Reading, MA, 1973.
[Le1] A. Ledet, On 2-groups as Galois groups, *Canad. J. Math.* **47** (1995), 1253–1273.
[Le2] A. Ledet, Embedding problems with cyclic kernel of order 4, *Israel J. Math.* **106** (1998), 109–131.
[Me] A. Merkurjev, On the norm residue symbol of degree 2, *Soviet Math. Dokl.* **24** (1981), 546–551.
[Mi1] I. Michailov, Some groups of orders 8 and 16 as Galois groups over $\mathbb{Q}$, *J. Number Theory,* to appear.
[Mi2] I. Michailov, Embedding problems with cyclic 2-kernel, *Israel J. Math.* (submitted).
[MZ] I. Michailov and N. Ziapkov, Embedding obstructions for the generalized quaternion group, *J. Algebra* **226** (2000), 375–389.